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What was Frege's notion of analyticity, and is it an improvement on Kant's?

Introduction

Mathematicians use systems of inference in order to further mathematical knowledge, but when does one of these inferences necessitate some sort of extra knowledge and when does the result simply ‘fall out’? Given a mathematical statement, will I be able to derive its conclusion from the premises using logic alone or will I need some extra intuition? Kant created the analytic-synthetic distinction which began to address these questions. This distinction was somewhat basic and was later expanded upon by Frege as part of his attempt to describe which areas of mathematics could be reduced to logic. This essay will define and analyse both concepts in order to determine whether Frege’s notion was an improvement on Kant’s. The conclusion of this essay will determine which notion best describes mathematical epistemology[1], this is a question which potentially influences the debate on mathematical ontology. Do mathematical objects exist independently of the mathematician? If so, how do we attain knowledge of them and how do we justify the inferences we make about them? First I will introduce Kant's notion and evaluate the main criticisms. Then I will describe Frege's notion and compare the merits and failings of each. I will conclude that the classification

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1 The epistemology of mathematics is the study of mathematical knowledge; concerning what we consider to be mathematical knowledge, how we can obtain it and how we may justify our beliefs of mathematical statements.

described by Frege is an improvement on Kant's as it captures more mathematical statements and is much closer to describing the epistemology of mathematics.

Preliminaries

First, I will consider Kant's notion of analyticity. For the purposes of this essay it will be necessary to find a common understanding of Kant's and Frege's notion. Proops (2005: pp. 588-612) gives a clear outline of the four possible characterizations of Kant's version of analyticity. I will use the notion of 'containment' described by Proops as it is this characterization which is most concerned with epistemology; it suggests that we can know some statements by virtue of their structure and by understanding the meaning of the concepts involved. For simplicity, I will first restrict attention to affirmative propositions. These are propositions that state an object *does* have a given property or that there *does* exist an object with a given property. Also, given that Frege only used the distinction for true propositions I will assume the same of Kant. First we must note that Kant's sense of logic was based on syllogism. Syllogism is a type of deductive reasoning which forms arguments from two premises and deduces a conclusion. An example being:

All Squid are Cephalopods
Some Invertabrates are Squid
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Some Invertebrates are Cephalopods

Kant's analytic-synthetic distinction is only concerned with the premises involved in syllogistic reasoning. These types of premises are subject-predicate statements; of which, if we are only to consider affirmative propositions, there are two kinds:

1. 'All S are P'
2. 'Some S are P'

Where S is the subject and P is the predicate. An example of type 1 would be 'All trees are tall', and of type two 'Some leaves are brown'.

Kant's Notion

In his Prolegomena, Kant describes analytic judgements as ‘those that merely spell out what’s already there, adding nothing to the content of the knowledge’ (Kant 1783: p. 7). These sorts of judgements are ones where ‘the predicate B belongs to the subject A as something that is (covertly) contained in this concept A’. (Kant 1781: p. 130). So, we may consider the statement ‘All Sheep are Four-legged’ to be analytic, as the concept of being 'Four-legged' is contained in the concept of 'Sheep'. According to Kant, concepts contain parts so we can analyse judgements in this way by asking if the predicate is a part contained in the subject. Hence, analytic judgements are a priori knowable - we can arrive at them through reasoning alone without the need for exterior experience. If a judgement is not analytic, then in order to justify it we must appeal to something more than just reasoning along with an understanding of the subject and predicate concepts. This type of judgement is classified as synthetic. Hence synthetic judgements are ones that expand our knowledge; these can fall into two categories: synthetic a priori and synthetic a posteriori. The synthetic a posteriori require empirical facts in order to justify their truth. Empirical facts are truths that can only be derived from experience and sense perception, such as 'There are eight planets in our solar system'. The synthetic a priori need no such facts though in order to be considered synthetic they need something extra than what we find contained in the subject concept. We see that Kant's analytic-synthetic distinction has an epistemic sense; it describes how we may may first arrive at a judgement *and* how we may know the judgement to be true.

The Limits of Syllogism

The first and most obvious criticism of Kant's account of mathematical inference is that his distinction is confined to subject-predicate forms, which we know to be very limited. This is a very divisive limitation; we can quickly see that even some of the most basic mathematical judgements are not captured as syllogisms do not allow for the quantifiers ‘all’ or ‘some’ in the predicate place and do not allow for more complex inferences (Kenny 1995: p. 20). An example being 'Some natural numbers are divisible by all natural numbers less than ten and greater than zero'. Thus Kant's distinction is unable to give a complete account of mathematical epistemology since there are statements that it cannot categorise.

The Problem of Subjectivity

Secondly, there is an objection that whether a judgement is analytic or not is essentially subjective. The statement ‘All Squid are Cephalopods’ may be analytic or not to separate individuals, turning on each individual’s understanding of the concept of ‘Squid’. For one person the concept of ‘Squid’ may contain 'Cephalopod' and for another person it may not. The defence given by Proops is that this objection ‘misidentifies the relativity in question’ (Proops 2005: p. 597). What is relative to each person is the concept associated to each word, not what each concept contains. So in the above example, the two people are considering the concept they associate with the word ‘Squid’ but these concepts are in fact distinct. This argument seems to suggest either that concepts exist independent of the words expressing them or that there is a correct concept associated to each word and that one of the people in the above example is misinterpreting the word. In the former case, Kant's distinction survives as we can decide whether a statement is analytic or not once we have agreed on the concepts involved. In the latter case arises a problem; as how are we to know which concept should be associated to a given word? However, words, particularly in mathematics, are only symbolic. For example, if we swap the symbols 'even' and 'odd' then mathematics would be essentially unchanged, as mathematics proceeds by forming a consensus on which concept a symbol denotes. So, the problem of subjectivity is not detrimental to Kant's distinction.

Frege's Notion

Frege describes a proposition as analytic if there is a proof of the proposition which on tracing back through the proof from the conclusion to the initial premises ‘we come only on general logical laws and definitions’ (Frege 1884: §3). An analytic proposition should be possible to prove by only looking at the definitions of the concepts involved and applying logical laws to them. As described by Dummett, ‘A statement is analytic in Frege’s sense if it is the definitional equivalent of an instance of a provable formula’ (Dummett 1991: p. 29).

What are considered to be initial premises, general logical laws and definitions? The initial premises are the assumptions that a judgement is based upon. If I postulate $'P'$ and $'P \rightarrow Q'$ and derive $'Q'$ then $'P'$ and $'P \rightarrow Q'$ are the initial premises of this judgement. If I cannot improve on the deduction by proving my initial premises then the initial premises are unprovable truths which Frege classifies in two ways: general laws and a posteriori facts - Frege refers to both of these as ‘primitive truths’. The propositions considered by Frege were only those of the type where the initial premises were primitive truths. General laws are, as described by Frege, those laws that ‘neither need nor admit of proof’, they don’t necessitate a proof and are, in fact, impossible to prove but they also must not lead to contradiction. We can think of general logical laws as logical rules of inference which are considered to be true regardless of context such as Modus Ponens:

$$\frac{P,\ P \rightarrow Q}{Q}$$

These general logical laws are a subclass of general laws. To see how Frege intended to distinguish general *logical* laws from general laws it will be useful to consider why Frege believed (Euclidean) geometry to be synthetic. It is possible to posit the negation of one or more of the axioms of Euclidean geometry and work within this new geometry without coming upon contradiction. For example, we can consider geometries where the parallel postulate does not hold and not arise at a contradiction. Hence, the parallel postulate is not universally true and thus the truths derived from it are synthetic. This shows that the general laws of Euclidean geometry, the axioms, are independent of general logical laws. So, we can see that a general logical law is supposed to be a law which must always be true in any context and that the negation of a general logical law is always contradictory/false. As I shall discuss, Frege gives no satisfying account for what he means by a definition though his intention was that definitions should add no new information in the deductive steps.

Frege distinguishes synthetic truths simply as truths that are not analytic. Hence, every proof of a synthetic judgement must use a truth which is not of a general logical nature, i.e. an empirical fact or a general law whose truth is context-dependent. Again, we see that analytic judgements must be a priori knowable and synthetic judgements may be split into the synthetic a priori and a posteriori. A judgement which contains an empirical fact in its initial premises is synthetic a posteriori. Such a judgement may be 'I have eight fingers', as the justification of this statement relies on sensory experience. A judgement which contains general laws in its proof which are not general *logical* laws is synthetic a priori, the triangle angle sum theorem is an example of this as it relies on Euclid's parallel postulate.

The epistemic sense of Frege’s distinction diverges slightly from Kant’s. For Frege, the distinction concerns *only* how we may justify the truth of a true statement - this is how we *could* know the truth of a statement. For Frege how we come to believe and grasp the content of a statement may be distinct from how we justify that it is true and the former may also involve inferences that are not logically valid. This is part of Frege’s attempt to cleave psychology from logic, he would like to ‘separate the problem of how we arrive at the content of a judgement from the problem of how its assertion is to be justified’ (Frege 1884: §3). As an example, it may be possible to know an a priori truth a posteriori. Consider the proposition ‘If there are 5 people living in London then there is at least 1 person living in London’. This is a priori true, however, I may come to know of its truth by knowing of one person who lives in London and hence deduce that it is true a posteriori.

<u>How does the epistemology of Frege's notion compare with that of Kant's?</u>

Frege correctly notes that Kant’s notion of analyticity can only be applied to propositions of subject-predicate form, and so is very limited. Whereas, Frege’s new notion of analyticity covers all (provable) propositions that are expressible with Frege's more advanced logic. In fact, it is possible to apply his notion to any formal proof system and hence all (provable) mathematical propositions[2]. Further, in contrast to Kant, this notion manages to separate how we form a sentence with

2 Within a given theory

how we analyse a sentence and know it to be true. With regards to Kant's notion, our knowledge about the truth of a sentence is based on our understanding of the structure of a sentence and parts implicit in its components. Frege's notion is much more complex and allows us to concern ourselves with the justification of mathematical statements as given by mathematicians i.e. by proof. Consider the statement 'For any natural numbers a,b and p, if p is a prime and p divides ab then p divides a or p divides b'. This is a simple statement from number theory and, given a specific interpretation of the concepts of natural and prime number that we assume to be true, it is easy to see from reading the mathematical proof why the statement must be true. If it is possible to formalize the proof and give definitions in a sufficient way then this statement will be analytic, if we can do this for all such mathematical statements then Frege's notion of analyticity really describes how a mathematician justifies truth. So, Frege's account is much closer to the epistemology of mathematics and mathematical practice and this illustrates well why Frege's notion is an improvement.

Is Frege's Notion Exhaustive?

As with Kant, there are potential problems for Frege as to whether his distinction is able to capture all mathematical statements. We should first note that Frege's notion is an extension of Kant's. If we consider a proposition that is analytic in Kant's sense we can see that it is also analytic in Frege's. Consider the proposition: 'All Squid are Cephalopods'. With squid defined as 'Cephalopod Molluscs of the Genus Loligo', we can substitute this into the proposition:

‘All Cephalopod Molluscs of the Genus Loligo are Cephalopods’ - which is clearly tautologous, having the form:

$$\forall x(C(x) \wedge M(x) \wedge L(x)) \rightarrow C(x)$$

So, given that this definition is legitimate, the proposition is also analytic in Frege's sense. Frege believed that truth-value in mathematics is not subjective, so what makes a statement true is independent of our knowledge of its truth. Further, according to Dummett, Frege thought that ‘mathematical statements are true or false, independently of whether we do or can prove them’ (Dummett 1973: p. xxxviii). If this is the case then we may have true statements that we cannot prove, which would be problematic for Frege as these statements would not be captured by the analytic-synthetic distinction. However, as was a feature of the time, Frege believed all mathematical truths to be provable. So, this is at least consistent with Frege’s beliefs. Nonetheless, there are number-theoretic results which are not provable within second-order arithmetic, the system Frege was concerned with. So the question still stands as to how to classify these statements. Raatikainen describes a theorem of second-order arithmetic that is not provable within this theory and where ZFC must be used in order to prove it (Raatikainen 2013: §4.5). I think it would be reasonable to classify such statements as synthetic. In the above example we must extend to ZFC in order to prove the statement and this implies that we are using laws that are not of a general logical nature, i.e. the axioms of ZFC. With respect to the problem of the distinction being exhaustive Frege still comes out on top. His notion is able to cover all that Kant's is able to cover and could even be considered exhaustive if we consider statements that are unprovable in second-order arithmetic to be synthetic.

The Problem of Definition

According to Dummett the most serious flaw in Frege's characterisation is the failure to give a criterion for when a definition is correct (Dummett 1991: p.30). This is very important for Frege's distinction as without a rigorous way for choosing definitions we may be able to pick definitions that make a statement trivially analytic. For example, the statement 'Every natural number can be written uniquely as a product of primes' would become trivially analytic if we define natural number as 'a number that can be written uniquely as a product of primes', thus rendering the distinction meaningless. This is similar to the criticism of Kant that a statement may or may not be analytic depending on one's interpretation of the subject concept. Is it possible to account for this criticism in the same way? In an attempt to do so we may say that what is relative to each person is the definition associated to each word, not the thing it is defining. This explanation appears to be sufficient and also fit with Frege's understanding of mathematical objects - that they exist objectively, independent of the mathematician. From this viewpoint, we may say that the definition is just plain wrong; which neatly brings us to the original criticism. How do we know when the definition is correct? We could consider definitions to just be symbolic, allowing us to replace instances of the defined term with the definition. This type of abbreviation is a weak form of definition, which manages to sidestep the above criticism but fails to define the meaning of a mathematical term, in particular its *sense* (Dummett 1991: §3). The sense of an expression being 'that part of its meaning which is relevant to the determination of the truth-value of sentences in which the expression occurs' (Dummett 1973: p. 89). Hence, if we are to use definitions to say anything meaningful about the (objective) truth of mathematical statements it is imperative that in defining mathematical terms, and thus mathematical objects, that we capture their sense. The idea of definition is fundamental to Frege's analytic-synthetic distinction though the problem remains as to how we are to know a definition to be correct.

Conclusion

We have seen that Kant's characterization is very limited, not expanding beyond the realms of syllogism, whereas Frege's comes much closer to being an exhaustive classification of mathematical statements. Kant's distinction may allow for subjectivity to creep in, though this argument doesn't appear to be valid. Here though arises a problem for Frege where he is unable to give a satisfying account for definitions; if we agree that they mustn't be subjective then there must be some criteria to identify when they are objectively correct. This problem may not be insurmountable, though how one decides to overcome this problem may have detrimental effects on the applicability of Frege's classification. The criticisms of both Kant and Frege are therefore very similar, with varying degrees of validity and importance. However, it is Frege that holds more epistemological weight. Mathematical truths are justified by rigorous proof and the method of proof is also how mathematical knowledge can be attained. Hence, any distinction regarding the epistemology of mathematics must account in some way for proof. This is what Frege's distinction attempts to do and, although not without problems, I must conclude that it is an improvement on Kant's distinction.

<u>Word Count: 3177</u>

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