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What was Hilbert's programme? Describe and critically evaluate what you take to be the most powerful objection to it

Introduction

Hilbert's programme was an attempt to establish the foundations of mathematics on the surest possible footing, relying on the certitude of the most basic mathematical objects. These objects are the subject of finitary arithmetic. Hilbert hoped that any branch of mathematics could then be validated by this solid base. This would therefore rid mathematics of inconsistency and ensure the ultimate reliability of the mathematical method. These extended branches of mathematics would be content-less and proceed via rule driven manipulations as in a game, judging the usefulness of each theory by its applicability to the sciences. Unfortunately, Gödel published a result which would prove fatal for the programme as conceived by Hilbert.

First, I will outline Hilbert's programme and its underlying philosophy. After this I will explain Gödel's Second Incompleteness Theorem and how Hilbert's programme fails due to this result. Then I will consider Gentzen's Theorem and possible revisions of the idea of finitary arithmetic. I will conclude, however, that even with such revisions Hilbert's Programme fails to provide a solid mathematical foundation.

Finitism

The basis of Hilbert's Programme is finitary arithmetic. This can be understood as elementary arithmetic but which avoids use of the infinite. This was an important point for Hilbert as, though Hilbert maintained that the infinite is an "indispensable notion" which has become very useful in mathematical thought, it has no place in the physical reality around us; "reality is finite" (Hilbert 1925: p. 372) and as a result intuition of infinite objects is not possible. For Hilbert, finitary mathematics is contentual - it has content and the propositions of finitary arithmetic can be interpreted. The content of such propositions are numerals such as '1', '1 1 1 1' and '1 1 1 1 1 1'. These numerals are not written to express anything about the physical forms of the numerals on the paper. It is best to imagine them symbolizing abstract versions of themselves. Hence, there is a level of abstraction present, though it is one that, according to Hilbert, is so natural to humans that it is necessary for humans to think at all. What will be important here is how we come to knowledge of finitary arithmetic. Hilbert claimed that the attainment of finitary knowledge happens intuitively. This is intuition in the Kantian sense. Intuition is profoundly intertwined with perception, linked to how we attain knowledge of the world around us. I may intuit, for instance, that my stool has three legs. There is also pure intuition, which allows us to consider the "forms of possible empirical intuitions" (Shapiro 2000: p. 81). We can thus intuit abstract objects that we do not see, such as an equilateral triangle. Pure intuition also allows us a faculty of mental construction – to the intuited equilateral triangle I may construct a line from the

centre of one line to the angle opposite it. In a similar way we can intuit things about finitary objects and come to conclusions about finitary arithmetic such as 'I I' + 'I I I' = 'I I I I I'.

Propositions such as '1+4=5' or '2^9=4*2^7' are therefore finitary (where we take 2 to be an abbreviation for 'I I' etc.). Though such propositions are only concerned with finite entities, these entities may become far too large for a human or even computer to calculate in a reasonable amount of time. However, what is important is that these propositions remain in the realm of finite objects and that there is an algorithm that could theoretically check the validity of the propositions in a finite amount of time. The truths of finitary arithmetic are thus reliable and where "contradictions and paradoxes arise only through our carelessness" (Hilbert 1925: p. 376). Hilbert also allowed for propositions of algebra to be considered finitary where there was an indication of the contentual interpretation of the symbols used i.e. where the the symbols are used to communicate numerals. This can also be extended to propositions with quantifiers, with conditions so that we avoid speaking of the infinite. Basically, an existential statement must assert the existence of a number with a certain property from a finite set, i.e. the quantifier must be bounded so that it is possible to check the proposition in a finite amount of time.

Finitary arithmetic is thus something that is most basic to mathematics and also something we can attain knowledge of in the most natural way possible, via intuition. This then provides mathematics with the surest possible footing.

Formal Mathematics

Now that we have a secure base for mathematics, how do we account for the mathematics not contained in finitary arithmetic? This is attained by attaching what Hilbert calls 'ideal mathematics'. In contrast to finitary arithmetic, the propositions of ideal mathematics have no content and are essentially meaningless (unless they correspond to finitary statements e.g. $a+b=b+a$). For example, group theory is an ideal branch of mathematics. When we add the symbol ' \circ ' to number theory to form group theory, we ascribe rules to which the operation ' \circ ' must follow but the propositions of group theory have no meaning in themselves. In Hilbert's words, "contentual inference is replaced by manipulation of signs according to rules" (1925, p. 381).

The Programme

The first step in Hilbert's programme was to sufficiently formalize the branches of mathematics into axiomatic theories. This involves creating a strict formal language which allows all mathematical propositions to be represented as formulas and then describing a proof theory which allows the mathematician to make inferences. A formal theory is then just a collection of sentences which are considered true within the given theory. In Hilbert's words, mathematics then becomes an "inventory of formulas" (Hilbert 1927: p. 465). Of these formulas there are those which provide the basis on which that branch of mathematics sits upon, the axioms, and those which can be proved by the theory, its theorems.

Naturally, Hilbert used the logical calculus of the time, forming sentences using ‘and’, ‘or’, ‘not’, ‘implies’ and the existential and universal quantifiers. Each branch of mathematics would then have a different language, but these logical connectives would be at their base, thus allowing for a formal proof theory. A proof of the statement B is then a finite string:

$$\frac{A \quad A \rightarrow B}{B}$$

Where A and A→B are axioms or are the last formula of an inference occurring earlier in the proof, i.e have already been proved. In this way, ideal mathematics can proceed via the manipulation of symbols according to definite rules. Also, such formal proofs are themselves finite strings of symbols and so are intuitable objects. We can intuit them in their totality and check their validity in a finite amount of time, hence “formal systems themselves now come under the purview of *finitary arithmetic*” (Shapiro 2000: p. 164). This is very important for Hilbert’s aims, as it may allow us to use finitary arithmetic to prove results about ideal theories.

Requirements

We now have branches of mathematics covered by axioms and rules of deduction. This all seems fitting for a modern day mathematician. However, we need to be careful about how we construct these ideal branches, as we may create

useless theories. Firstly, Hilbert gives special status to the axioms of finitary arithmetic, they are the surest mathematical truths we can obtain, hence any theory we create must not contradict them. Secondly, Hilbert's most crucial requirement is that any formal theory should be consistent. A theory, T , is consistent if there is no formula, θ , such that $T \vdash \theta$ and $T \vdash \neg\theta$. An inconsistent theory can prove anything, hence a proof given in an inconsistent theory is useless. For example, say we take an inconsistent theory, F , which we want to use to model fluid dynamics. From F we could derive that the energy of the system is 10 joules, but we could also arrive at the result that the energy of the system is 100 joules.

However, to accomplish the main purpose of Hilbert's programme Hilbert desired that the consistency of each formal theory be proved using only finitary means. This would mean that we can be as sure as possible of the consistency of the theory. Here lies the tip of a problem for Hilbert as consistency is an elusive property and is not trivial to prove.

The Objection

The most powerful objection to Hilbert's programme comes via Gödel's Second Incompleteness Theorem (GIT2) and a suitable interpretation of 'finitary'. The objection is that any sufficiently powerful and consistent formal theory cannot prove its own consistency, hence Hilbert's requirement for consistency is

untenable. Here, sufficiently powerful means that the theory can do basic arithmetic.

The statement of the theorem as given by Raatikainen (2013: §3.1) is:

“Assume F is a consistent formalized system which contains elementary arithmetic. Then $F \not\vdash \text{Cons}(F)$.”

Here 'Cons(F)' is the predicate stating that F is consistent and 'containing elementary arithmetic' means that F contains Robinson Arithmetic (ROB). ROB is Peano Arithmetic – the standard theory of arithmetic – without the induction axiom. To say that F contains ROB means that every statement provable in ROB is also provable in F .

How exactly does this affect Hilbert's programme? Well, given a formal theory, T , that contains ROB, we ask does T contain finitary arithmetic? It is generally accepted, given the basic nature of finitary arithmetic as described by Hilbert, that ROB does contain finitary arithmetic (if ROB is not sufficient, then we can add the induction axiom and consider Peano Arithmetic but the result is much the same). Thus, by GIT2 T cannot prove its own consistency. As T contains finitary arithmetic anything provable by finitary arithmetic is also provable by T . Hence, if T cannot prove that T is consistent then neither can finitary arithmetic. We are then left in the position that any theory strong enough to do ROB does not satisfy

Hilbert's requirement for consistency. Thus Hilbert's consistency requirement is unachievable, removing the possibility of placing mathematics on the sure foundations of finitary arithmetic.

This isn't the end of the discussion however. GIT2 relies on a specific sentence for expressing consistency. It may be true that F cannot prove this specific sentence, but are there other sentences expressing consistency that F can prove? This would mean that F can in fact prove its own consistency and leave the possibility that finitary arithmetic can also prove the consistency of F . Also, it may be the case that finitary arithmetic isn't contained in ROB and that there exist finitary means at our disposal that go beyond basic arithmetic, this would dodge Gödel's trap as though a theory F may not be able to prove its own consistency there would still be the possibility that finitary arithmetic can.

In order to address these questions I will describe Gödel's results in more detail. It will also be instructive to understand better why containing arithmetic is a necessary condition. In order to consider ' $F \vdash \text{Cons}(F)$ ', we must be able to write ' $\text{Cons}(F)$ ' as a sentence in the language of F . Gödel devised an ingenious technique called Gödel numbering which is an algorithm for assigning natural numbers, in a unique way, to formulas in the language of F . As proofs in a formal system are just finite strings of formulas it is then possible to assign a natural number, again uniquely, to every proof in F . We can then assert the existence of a formula in the language of F which defines a predicate expressing that a natural

number x is associated to the proof of a sentence θ . With existential quantification this gives us the provability predicate $\exists xPr(x, [\theta])$. Where $[\theta]$ is the Gödel number of the sentence θ . This predicate states that there exists a proof of θ in F . We can now formalize 'Cons(F)' as $\neg\exists xPr(x, [\theta])$ where we take θ to be an inherently contradictory sentence. To speak of $[\theta]$ F must contain some amount of number theory, also for example, Gödel numberings use ideas of primality, and it turns out that Robinson Arithmetic is sufficiently strong to do this.

Does it matter how consistency is formalized?

Above I described that consistency is formalized as the sentence stating that there is no proof of a given contradiction. However, an explicit construction for the formula defining the provability relation is not given. This poses potential problems for Gödel's Theorem. In fact, given a formula defining the provability relation we can construct new formulas that also define the provability relation but render the theorem false or even trivially true. However, as described by Raatikainen (2013: §3.2), it is possible to stipulate conditions on the formalization of the provability predicate such that GIT2 holds in the desired way. These conditions also ensure that the provability predicate behaves like the natural notion of provability we would expect so I will instead consider the second point.

Can all finitary means be formalized in ROB?

If finitary means are not confined to basic arithmetic then we may be able to use a theory that is considered finitary and is also able to prove the consistency of theories such as Peano arithmetic (PA). This may allow Hilbert's Programme to survive the above objection. For a well-established example, I will look at Gentzen's Theorem. Gentzen (1936) proved the consistency of PA from within Primitive Recursive Arithmetic (PRA) with the addition of transfinite induction¹ up to ε_0 ². PRA is the logical foundation of arithmetic as given by Skolem (1923) which is similar to PA (and hence ROB), however it does not allow unbounded quantifiers. Though Takeuti (1987: p. 90) claims that Hilbert's finitary arithmetic could be considered as "that which can be formalized in primitive recursive arithmetic", he does give a compelling account of how Gentzen's proof using transfinite induction may be considered as finitary.

1 This is best understood as normal mathematical induction which allows 'jumps' up to larger ordinals

2 ε_0 is a particular countably infinite ordinal

Here is a brief outline of the idea of the proof as described by Takeuti:

- Each proof in PA can be associated to an ordinal less than ϵ_0
- ϵ_0 is shown to be accessible³, and so any ordinal less than ϵ_0 is also accessible
- It is shown that for any contradiction in PA the ordinal associated to its proof is not accessible
- Hence no contradictions are provable in PA
- Hence PA is consistent

It is argued that such a proof then relies only on “concrete sequences of concrete figures, concrete operations on them, concrete operations on concrete operations, and so on” (Takeuti 1985: p. 257).

How can the objects and operations involved be considered 'concrete'? Well, we may consider how arbitrarily large sequences of numerals can be considered concrete though they are not written on paper. We begin with a concrete figure '1'. Then we take concrete operations that explain how to create new expressions, in this case: if 'a' is a numeral then so is 'a 1'. We then say that

³ An ordinal is accessible if any strictly decreasing sequence beginning with the ordinal is finite

this defines numerals, i.e. anything obtained from this procedure is a numeral and all numerals can be obtained via this procedure.

In a similar way we can have an inductive definition of ordinals, as given by Takeuti (1987: p. 90):

“ O1 0 is an ordinal.

O2 Let μ and μ_1, \dots, μ_n be ordinals. Then $\mu_1 + \mu_2 + \dots + \mu_n$ and ω^μ are ordinals.”

This definition then captures all ordinals less than ε_0 and may be used to claim that any such ordinal defined in this way is a concrete object.

The only controversial part of Gentzen's proof of the consistency of PA then boils down to the legitimacy of concrete manipulations of concrete objects. These objects express notions of actual infinite entities and so could not be admitted in a finitary proof in the strict sense. However, it appears possible to extend the notion to allow for such concrete operations and that such a revised notion would still be a faculty of intuition. After all, the level of abstraction required for such definitions and arguments appears to be very similar to that required to do calculations on large natural numbers. Moreover, Tait describes that a function, f , from A to B could be understood from a finitary point of view as “a specific procedure for defining [an element of] B from an arbitrary [element of] A ” (Tait 1981: p. 528). This is analogous to giving a concrete operation which defines how to construct new

ordinals from ones already given. Hence, concrete operations and objects could be considered finitary.

With this revised version of finitism could Hilbert's programme then proceed as desired? Could we prove, say, the consistency of ZFC? Such a proof has not been given and would certainly require induction up to an ordinal larger than ε_0 . Also, a crucial element was that finitary arithmetic is as sure a grounding as is possible for mathematics and that any inferences made using finitary arithmetic are as certain as humanly possible. Though it may be possible to consider arguments using transfinite induction as in some way concrete, the epistemic strength of these arguments is not the same as that conceived by Hilbert as they are no longer contentual. This is an important metaphysical point. Above, I gave inductive definitions of numerals and ordinals. However, such definitions are only constructive and not actually definitional of the objects. Hence, the arguments involving them are not based on "contentual inference".

Further, it may even be the case that primitive recursive arithmetic with transfinite induction up to ε_0 is inconsistent. Hence, we cannot be certain that a revised version of finitary arithmetic allowing for transfinite induction immunizes Hilbert's Programme from Gödel's Theorem.

Conclusion

Within standard proof systems and with a strict understanding of finitary, I have shown that Gödel's Second Incompleteness Theorem proves that the consistency of useful mathematical theories is unable to be proven by finitary means alone. This leaves Hilbert's conception of mathematics without a foundation. Without being able to verify the consistency of a theory using methods that appear infallible an element of doubt is cast over the worth of any formal theory. Even if we extend our definition of finitary to include Gentzen's methods of transfinite induction we are still left with open questions regarding the consistency of strong mathematical theories.

Formal mathematics has however proceeded somewhat regardless and Hilbert's ideas live on. Formal theories such as ZFC are used and they are generally assumed to be consistent as no contradictions have yet been found. It would be ideal to find a consistency proof of ZFC, though this would likely be in a system stronger than ZFC thus denying the epistemic certainty desired by Hilbert. It appears then that there is no absolutely sure way of proving the consistency of formal mathematical theories, leaving us in the disconcerting position of having to just put faith into the consistency of any theories we use.

Word Count: 3185

Bibliography

Gentzen, G. 1936. The consistency of arithmetic. English Translation in *The collected papers of Gerhard Gentzen*. Amsterdam : North-Holland Pub. Co. 1969.

Hilbert, D. 1925. *On The Infinite*. English Translation in van Heijenoort, J. *From Frege to Gödel*. Massachusetts: Harvard University Press. 1967.

Hilbert, D. 1927. *The Foundations of Mathematics*. English Translation in van Heijenoort, J. *From Frege to Gödel*. Massachusetts: Harvard University Press. 1967.

Kleene, S.C. 1952. *Introduction to meta-mathematics*. Groningen : North-Holland Pub. Co.

Raatikainen, P. 2013. Gödel's Incompleteness Theorems. *The Stanford Encyclopedia of Philosophy*. [Online]. Spring 2015 Edition. [Accessed 20 April 2016]. Available from: <http://plato.stanford.edu/archives/spr2015/entries/goedel-incompleteness>

Shapiro, S. 2000. *Thinking About Mathematics*. Oxford: Oxford University Press.

Skolem, T. 1923. *The foundations of elementary arithmetic established by means of the recursive mode of thought, without the use of apparent variables ranging over infinite domains*. English Translation in van Heijenoort, J. *From Frege to Gödel*. Massachusetts: Harvard University Press. 1967.

Tait, W. 1981. Finitism. *The Journal of Philosophy*. **78**(9), pp. 524-546

Takeuti, G. 1985. Proof Theory and Set Theory. *The Present State of the Problem of the Foundations of the Mathematics*. **62**(2), pp. 255-263

Takeuti, G. 1987. *Proof Theory*. 2nd ed. Amsterdam: North-Holland Pub. Co.

Zach, R. 2006. Hilbert's Program Then and Now. *Philosophy of Logic*. **5**, pp. 411-447.

Zach, R. 2003. Hilbert's Program. *The Stanford Encyclopedia of Philosophy*. [Online]. Spring 2016 Edition. [Accessed 20 April 2016]. Available from: <http://plato.stanford.edu/archives/spr2016/entries/hilbert-program/>