Introduction to O-minimality

Name: Adam Richard-Bollans

Student Number: 200454459

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Supervisors: Dr Immanuel Halupczok & Dr Vincenzo Mantova

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Abstract

An introduction to o-minimality (sometimes referred to as *order*-minimality) is provided in this paper. An o-minimal structure is a model-theoretic structure containing a dense linear order where the subsets (of the universe of the structure) definable with parameters are finite unions of intervals and points. The paper begins with the definition of an o-minimal structure, giving two key theorems and then developing to look at further properties of specific o-minimal structures. The two key theorems are the monotonicity theorem, due to Pillay and Steinhorn [1], and cell decomposition, due to Knight, Pillay and Steinhorn [2]. The paper is largely based on the principal text on o-minimality, Lou van den Dries' 'Tame Topology and O-minimal Structures' [3]. The basic ideas of o-minimality are introduced in chapter one, where the set-theoretic and model theoretic definitions are given and the equivalence between them is proved. The model-theoretic definition of *definable* gives a powerful tool for deciding which sets belong to a given structure. The model theoretic notions should be easily comprehensible to anyone familiar with the basics of model theory and a grasp of first-order logic, also model theory features heavily in chapter four so some familiarity with model theory and first-order logic is recommended. The second chapter is devoted to monotonicity and cell decomposition. The monotonicity theorem states that any function definable in an o-minimal structure can be split into finitely many subintervals on which the function is 'well-behaved'. The cell decomposition states that given any finite set of definable sets, there is a partition of the ambient space into *cells* which also partitions the given sets. These two theorems may not at first seem particularly strong or surprising, however the following chapters will to some extent highlight how useful they are. The third chapter begins to look more closely at the topology of the ambient space induced by the ordering and considers properties of a restricted class of o-minimal structures - expansions of ordered abelian groups. The level of topology required to understand this is very low and an aside is provided which should be sufficient. Chapter three also works inside some group theoretic structure, though the material can be understood without any knowledge of group theory. The main result of chapter three being *definable curve selection*, due to van den Dries, which shows that a point in the closure of a definable set is a limit point of some path in the set. Chapter four considers the exponential function in o-minimal structures, firstly considering the exponential function on the real line and then defining the exponential function on an arbitrary dense linearly ordered set. The main theorem of this chapter is the growth dichotomy, due to Miller [4], which states that for every o-minimal structure that contains an ordered field either every definable function is bounded by a *power function* or the exponential function is definable. Chapter four involves group homomorphisms but these can be easily understood by any model theorist as structure preserving maps. However, some familiarity with fields is certainly necessary throughout this chapter. Finally, some applications of o-minimality are briefly discussed in chapter five. This paper is mainly concerned with properties of arbitrary o-minimal structures, however a large area of research into o-minimality is devoted to finding out which structures are o-minimal. The beauty of this subject is that the o-minimality condition is an excellent restriction. O-minimal structures have some very nice properties, yet the class of o-minimal structures is still very large. Hence there are many useful and well researched structures which are o-minimal, for example the semialgebraic sets - used extensively in algebraic geometry - and, as deduced from a result due to Wilkie [5], the real ordered field with exponentiation. Also, the real ordered field with restricted analytic functions [6] and the real ordered field with exponentiation and restricted analytic functions [7] are o-minimal. The paper is thus aimed at mathematicians who are familiar with model theory and first-order logic and would like to gain some familiarity with o-minimality from a purely model-theoretic perspective or to apply o-minimality to other areas of maths, due to the nature of o-minimality this may include students interested in topology, group theory, combinatorics, algebra or analysis.

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1. Introduction to o-Minimal Structures

1.1. Introduction

In this chapter we build up the necessary framework to look at the ideas fundamental to the concept of o-minimality. This begins with a look at what is meant by a structure on a given set. We see that structures have some nice properties and that this notion of structures is closely linked with the notion of definability from model theory. We then define the condition for a structure on a set to be o-minimal and and for a model-theoretic structure to be o-minimal. We then briefly describe some properties of o-minimal structures as well as some important examples of o-minimal structures.

1.2. Structures

We look at a structure as a set-theoretic object, more specifically a structure is a nonempty collection of sets that is closed under certain operations. At first the given definition of a structure may seem somewhat arbitrary and not be immediately clear why such objects are important. However, as I will show, the way that such structures are defined means that we can deduce if a set is contained in the structure if there is an appropriate first-order defining it (in a model-theoretic sense).

Before introducing the definition of a structure it will be necessary to define what is meant by a boolean algebra of subsets of a given set X.

Definition 1.2.1. *[3, p. 12] A* ***boolean algebra of subsets of a set*** X *is a non-empty collection* $\mathcal{C}$ *of subsets of* X *such that if* $A, B \in \mathcal{C}$ *then* $A \cup B \in \mathcal{C}$ *&* $X - A \in \mathcal{C}$

Van den Dries notes that for a boolean algebra of subsets of X, $\mathcal{C}$, $X \in \mathcal{C}$, $\emptyset \in \mathcal{C}$ and if $A, B \in \mathcal{C}$ then $A \cap B \in \mathcal{C}$ [3, p. 12].

To see why this is the case, note that $X = (X - A) \cup A \in \mathcal{C}$ for some set $A \subseteq X$ (Such an A exists as $\mathcal{C}$ is nonempty). Further, $X - X = \emptyset \in \mathcal{C}$. Also, to see that $A \cap B \in \mathcal{C}$ note that $A \cap B = X - ((X - A) \cup (X - B))$.

Example 1.2.2. To see a trivial example: $\mathcal{C} = \{\emptyset, \mathbb{N}, \{x \in \mathbb{N} : x \text{ is even }\}, \{x \in \mathbb{N} : x \text{ is odd }\}\}$ is a boolean algebra of subsets of $\mathbb{N}$.

Definition 1.2.3. *[3, p. 13] A **structure** on a nonempty set R is a sequence $\mathcal{S} = (\mathcal{S}_m)_{m \in \mathbb{N}}$ such that for each $m \geq 0$:*

(S1) $\mathcal{S}_m$ is a boolean algebra of subsets of R^m

(S2) if $A \in \mathcal{S}_m$, then $R \times A$ & $A \times R$ belong to $\mathcal{S}_{m+1}$

(S3) $\{(x_1, ..., x_m) \in R^m : x_1 = x_m\} \in \mathcal{S}_m$

(S4) if $A \in \mathcal{S}_{m+1}$ then $\pi(A) \in \mathcal{S}_m$ where $\pi : R^{m+1} \to R^m$ is the projection map on the first m co-ordinates

Note Instead of saying that $\mathcal{S}$ is a structure on R we may also say that $(R, \mathcal{S})$ is a structure. Also, we say that a set A **belongs** to $\mathcal{S}$ if it belongs to $\mathcal{S}_m$ for some m.

The definition of a structure in fact ensures that any set A belongs to $\mathcal{S}$ if it is definable from the sets of $\mathcal{S}$. Van den Dries does not make explicitly clear why this is the case. For some intuition behind the conditions, in some sense we can think of $X - A$, $A \cup B$ and $A \cap B$ from (S1) as providing closure under negation, conjunction, and disjunction respectively and (S4) providing closure under existential quantification. In Lemma 1.3.7 we will see more clearly that if we are given some sets R &$B_i, i \in I$ with $\mathcal{S}$ being the smallest structure on R containing the B_i, then a set A is definable from the B_i if and only if A belongs to $\mathcal{S}$. In particular, if a structure $\mathcal{S}$ on R is given then consider the smallest structure containing all the sets of $\mathcal{S}$. This must be $\mathcal{S}$ as it is already a structure. Then, by applying Lemma 1.3.7 we will see that a set A is definable from the sets in $\mathcal{S}$ if and only if A belongs to $\mathcal{S}$. For this reason, when $\mathcal{S}$ is given, we call A **definable** if A belongs to $\mathcal{S}$.

It will be necessary to have some way of speaking about functions in relation to structures, in this way we will be able to classify functions. We do this by saying that a function belongs to a structure if its *graph* does. See the definition of the graph of a function below.

Definition 1.2.4. *[3, p. ix] Let $f : X \to Y$ be a function. The **graph** of f, $\Gamma(f)$, is defined to be the set*

$$\{(x, y) : x \in X \;\&\; y = f(x)\} \subseteq X \times Y$$

.

Terminology We say that f belongs to $\mathcal{S}$ if its graph, $\Gamma(f)$, belongs to $\mathcal{S}$.

1.2.1. Basic Properties of Structures

The following lemma gives some basic properties of structures that will be useful later on. These properties show how structures can be considered closed under certain operations.

Lemma 1.2.5. *[3, p. 13] Let $\mathcal{S}$ be a structure on R.*

(i) If $A \in \mathcal{S}_m$ and $B \in \mathcal{S}_n$, then $A \times B \in \mathcal{S}_{m+n}$

(ii) For $1 \leq i < j \leq m$ the set $\Delta_{ij} := \{(x_1, ..., x_m) \in R^m : x_i = x_j\}$ belongs to $\mathcal{S}$

(iii) Let $B \in \mathcal{S}_n$, and let $i(1), ..., i(n) \in \{1, ..., m\}$. Then the set $A \subseteq R^m$ defined by the condition

$$(x_1, ..., x_m) \in A \iff (x_{i(1)}, ..., x_{i(n)}) \in B$$

belongs to $\mathcal{S}$.

Proof. (i) Omitted. This can easily be seen from (S1) and (S2) so I will omit this proof.

(ii) Due to Van den Dries [3, p. 14]. Let $i, j \in \mathbb{N}$ with $1 \leq i < j \leq m$. Then $A = \{(x_1, ..., x_{j-i}) \in R^{j-i} : x_1 = x_{j-i}\}$ belongs to $\mathcal{S}$ by (S3). Then $\Delta_{ij} = R^{i-1} \times A \times R^{m-j-1}$ belongs to $\mathcal{S}$ by (i).

(iii) Let $B \in \mathcal{S}_n$. Then $R^m \times B \in \mathcal{S}_{m+n}$ by (i). Now,

$$\Delta_{i(1)1+m} \cap R^m \times B = \{(x_1, ..., x_m, y_1, ..., y_n \in R^m \times B : x_{i(1)} = y_1\}$$

belongs to $\mathcal{S}$ by (ii) and (S1), where $\Delta_{i(1)1+m} \subseteq R^{m+n}$. Repeating this process n-1 times for $\Delta_{i(2)2+m}, \Delta_{i(3)3+m}\cdots$ we obtain the set

$$C = \{(x_1, ..., x_m, y_1, ..., y_n) \in R^m \times B : x_{i(1)} = y_1, ..., x_{i(n)} = y_n\}$$

which again belongs to $\mathcal{S}$ by (ii) and (S1). We can apply the projection map to C to get

$$\{(x_1, ..., x_m, y_1, ..., y_{n-1}) \in R^{m+n-1} : x_{i(1)} = y_1, ..., x_{i(n-1)} = y_{n-1} \\ \& \; x_{i(n)} = y_n \text{ for some } y_n \text{ such that } (y_1, ..., y_n) \in B\}$$

which belongs to $\mathcal{S}$ by (S4).

Applying the projection map $n-1$ more times we obtain the set

$$\{(x_1, ..., x_m) \in R^m : x_{i(1)} = y_1, ..., x_{i(n)} = y_n \\ \text{for some } y_1, ..., y_n \text{ such that } (y_1, ..., y_n) \in B\}$$

which belongs to $\mathcal{S}$ by (S4). This is the required set, A.

□

1.3. Definability

In this section we formalize how to deduce membership of a structure using the idea of definability from model theory. This makes the process much simpler, removing the need to directly prove membership from (S1-4).

The definition given below briefly formalizes the notion of a model-theoretic structure. A language $\mathcal{L}$ is a collection of relation, function and constant *symbols* and an $\mathcal{L}$-structure gives a semantic meaning to the symbols in $\mathcal{L}$, interpreting the symbols as relation, functions and constants within the structure.

Definition 1.3.1. *Fix a language, $\mathcal{L}$, with relation symbols, T_i, function symbols f_j and constant symbols c_k, indexed by sets I, J, K respectively; an* ***model-theoretic structure*** *is a tuple*

$$\mathcal{M} = \langle M, T_i(M), f_j(M), c_k(M)\rangle$$

Where $T_i(M), f_j(M), c_k(M)$ denote the interpretations of the relations, function and constant symbols in $\mathcal{M}$. We say that $\mathcal{M}$ is an $\mathcal{L}$-structure and call M the universe of $\mathcal{M}$.

Given a subset $C \subseteq M$ we denote the language obtained by extending it with constant symbols from C by $\mathcal{L}_C$ and we denote the $\mathcal{L}_C$-structure obtained by adding constants from C by $\mathcal{M}_C$.

Also, it should be noted that we always have a symbol for equality and the interpretation of this symbol is the 'usual' notion of equality.

Example 1.3.2. $Z = \langle \mathbb{Z}, +, - \rangle$ is a model in the language with one binary and one unary function symbol.

Let $C = \{0, 1, 2\}$. Then $Z_C = \langle \mathbb{Z}, +, -, 0, 1, 2 \rangle$.

The idea of definable set of a model is that there is a formula which defines it i.e. there is a formula which is satisfied in the model by all elements of the set and only satisfied by elements of this set. This is made more formal below.

Definition 1.3.3. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure as above and $B \subseteq M$. We say that $X \subseteq M^n$ is **definable over** B **in** $\mathcal{M}$ if there is an $\mathcal{L}$-formula*

$$\varphi(x_1, ..., x_n, y_1, ..., y_k) \ \& \ b_1, ..., b_k \in B$$

such that for any $a_1, ..., a_n \in M$:

$$\mathcal{M} \models \varphi(a_1, ..., a_n, b_1, ..., b_k) \iff (a_1, ..., a_n) \in X$$

In particular, we say that $X \subseteq M^n$ is definable in $\mathcal{M}$ if X is definable as above but with $B = \emptyset$. If $B = M$ we say that X is definable using constants.

Example 1.3.4. (1) Let $R = \langle \mathbb{N} - \{0\}, \times \rangle$ where $\times$ is the usual interpretation of multiplication on the natural numbers. Then the formula $\varphi(x) : -\exists y : 2 \times y = x$ defines the set of even natural numbers greater than zero in R.

(2) Let $R = \langle \mathbb{R}, < \rangle$ where $<$ is the usual interpration of the ordering of the real numbers. Then the only definable subsets of $\mathbb{R}$ in R are $\emptyset$ and $\mathbb{R}$.

The subsets of $\mathbb{R}$ definable in R over $\{1\}$ are $\{x \in \mathbb{R} : x < 1\}$, $\{x \in \mathbb{R} : 1 < x\}$ and $\{1\}$.

Definition 1.3.5. *[3, p. 22] Given a model-theoretic structure $\mathcal{R} = \langle R, T_i, f_j, c_k \rangle$, we let Def($\mathcal{R}$) be the smallest set-theoretic structure on the set R that contains each relation T_i, function f_j and constant c_k.*

Example 1.3.6. Let $R = \langle \mathbb{Z}, <, ^2, 0, 1 \rangle$. Then $\{(x, y) \in \mathbb{Z}^2 : x < y\}$ (from $<$), $\{(x, x^2) \in \mathbb{Z}^2 : x \in \mathbb{Z}\}$ (from ^{2}), $\{0\}$, $\{1\}$ and $\{(x, x) \in \mathbb{Z}^2 : x \in \mathbb{Z}\}$ (from $=$) are all contained in Def(R). Also, all sets of the form $\{(x_1, ..., x_m) \in \mathbb{Z}^m : x_1 = x_m\}$ are in Def(R).

Moreover, all the above sets and combinations of the above sets due to operations from (S1),(S2) and (S4) are in Def(R) and these are the only sets in Def(R).

Lemma 1.3.7. *Given a language, $\mathcal{L}$, let $\mathcal{R} = \langle R, T_i, f_j, c_k \rangle$ be an $\mathcal{L}$-structure. Then for any $n \in \mathbb{N}$ and any subset $A \subseteq R^n$*

$$A \in \textit{Def}(\mathcal{R}) \iff A \textit{ is definable in } \mathcal{R}$$

Proof. Let $Def(\mathcal{R}) = \mathcal{S} = (\mathcal{S}_m)_{m \in \mathbb{N}}$
I will first consider the forward, $\Rightarrow$, direction. Let $A \subseteq R^n$ and $A \in$ Def($\mathcal{R}$). As Def($\mathcal{R}$) is the smallest structure on $\mathcal{R}$ containing the relations, functions and constants then any set belonging to Def($\mathcal{R}$) must either be one of T_i, $\Gamma(f_j)$, $\{c_k\}$ or $\{(x_1, ..., x_n) \in R^n : x_1 = x_n\}$ or a finite combination of such sets from the operations induced by (S1), (S2) and (S4). Hence we can prove by induction on the 'complexity' of the set A.

Base Cases   Clearly if $A = T_i, \Gamma(f_j)$, or $\{c_k\}$ for some $i \in I, j \in J$ or $k \in K$ then A is definable in $\mathcal{R}$.

Suppose $A = \{(x_1, ..., x_n) \in R^n : x_1 = x_n\}$ for some $n \geq 1$, then $\varphi(x_1, ..., x_n) : x_1 = x_n$ defines A in $\mathcal{R}$.

Induction Hypothesis   Let $B, C \in Def(\mathcal{R})$ be definable and defined by the formulas $\varphi(x_1, ..., x_m)$ and $\theta(x_1, ..., x_k)$ respectively.

Split into cases according to the operations from (S1),(S2) and (S4). i.e. $A = B \cup C$, $A = R^n - B$ (from S1), $A = R \times B$, $A = B \times R$ (from S2) or $A = \pi(B)$ (from S4).

Must show that in either case that A is definable.

(S1) Let $n = m = k$. If $A = B \cup C$ then A is defined by the formula

$$\psi(x_1, ..., x_n) := \varphi(x_1, ..., x_n) \vee \theta(x_1, ..., x_n)$$

& if $A = R^n - B$ then A is defined by the formula $\psi(x_1, ..., x_n) := \neg\varphi(x_1, ..., x_n)$.

(S2) Suppose $n = m + 1$. If $A = R \times B$ then A is defined by the formula

$$\psi(x_1, ..., x_n) := \varphi(x_2, ..., x_n)$$

& if $A = B \times R$ then A is defined by the formula $\psi(x_1, ..., x_n) := \varphi(x_1, ..., x_{n-1})$.

(S4) Suppose $n = m - 1$. If $A = \pi(B)$. Then A is defined by the formula $\psi(x_1, ..., x_n) := \exists y : \varphi(x_1, ..., x_n, y)$.

This concludes the forward direction.

For the reverse direction, $\Leftarrow$, let A be defined by $\psi(x_1, ..., x_n)$. Will prove by induction on complexity of the formula.

Base Case Suppose $\psi(x_1, ..., x_n)$ is atomic. Let $i(1), ..., i(k) \in \{1, ..., n\}$.

For simplicity I will suppose the language is relational i.e. ignore function and constant symbols.

Then

$$\psi(x_1, ..., x_n) := T_b(x_{i(1)}, ..., x_{i(k)})$$

for some $b \in I$.

Now, for $a_1, ..., a_n \in R$

$$(a_1, ..., a_n) \in A \iff (a_{i(1)}, ..., a_{i(k)}) \in T_b$$

Thus, since T_b belongs to Def($\mathcal{R}$), A also belongs to Def($\mathcal{R}$) by Lemma 1.2.5(iii).

Induction Hypothesis Let $\theta(x_1, ..., x_m)$ and $\varphi(x_1, ..., x_k)$ define sets which belong to Def($\mathcal{R}$). Call these sets A_θ and A_φ.

Let $i_1, ..., i_{m+k} \in \{1, ..., n\}$. It must be shown that if $\psi(x_1, ..., x_n)$ is a boolean combination of $\theta(x_{i(1)}, ..., x_{i(m)})$ and $\varphi(x_{i(m+1)}, ..., x_{i(m+k)})$ then A belongs to Def($\mathcal{R}$).

Suppose $\psi(x_1, ..., x_n) := \theta(x_{i(1)}, ..., x_{i(m)}) \wedge \varphi(x_{i(m+1)}, ..., x_{i(m+k)})$. Let $a_1, ..., a_n \in R$.

Then

$$\begin{aligned}(a_1, ..., a_n) \in A &\iff \mathcal{R} \models \psi(x_1, ..., x_n)\\ &\iff \mathcal{R} \models \theta(a_{i(1)}, ..., a_{i(m)}) \wedge \varphi(a_{i(m+1)}, ..., a_{i(m+k)})\\ &\iff (a_{i(1)}, ..., a_{i(m)}) \in A_\theta \text{ and } (a_{i(m+1)}, ..., a_{i(m+k)}) \in A_\varphi. \ (*)\end{aligned}$$

Now, if $m = k$ let A'_φ be the set defined by

$$(x_{i(1)}, ..., x_{i(m)}) \in A'_\varphi \iff (x_{i(m+1)}, ..., x_{i(m+k)}) \in A_\varphi.$$

Then A'_φ belongs to Def($\mathcal{R}$) by Lemma 1.2.5(iii).

Hence by (*) $(a_1, ..., a_n) \in A \iff (a_{i(1)}, ..., a_{i(m)}) \in A_\theta \cap A'_\varphi$. (Note that $A_\theta \cap A'_\varphi$ belongs to Def($\mathcal{R}$) by (S1)).

Hence A belongs to Def($\mathcal{R}$) by Lemma 1.2.5(iii).

If $m \neq k$, suppose without loss of generality that $m < k$. Let $t = k - m$.

The set $B = A_\theta \times R \times ... \times R \subseteq R^{m+t}$ belongs to Def($\mathcal{R}$) by (S2).

Then $(a_1, ..., a_n) \in A \iff (a_{i(1)}, ..., a_{i(k)}) \in B$ and $(a_{i(m)}, ..., a_{i(m+k)}) \in A_\varphi$. Now we can repeat a similar argument as for the case where $m = k$ to see that again A belongs to Def($\mathcal{R}$).

This concludes the proof for the case

$$\psi(x_1, ..., x_n) := \theta(x_{i(1)}, ..., x_{i(m)}) \wedge \varphi(x_{i(m+1)}, ..., x_{i(m+k)}).$$

For brevity I will leave the cases for $\vee$, $\neg$, $\rightarrow$ and existential and universal quantification, though the idea of the proof of these is much the same.

$\square$

Now, due to the above lemma, if we are concerned with finding out if a given set A belongs to a given structure $\mathcal{S}$ we only need to find a formula defining it in the appropriate model. To do this, consider Def($\mathcal{R}$) where $\mathcal{R}$ is the model-theoretic structure $\langle R, T_i \rangle$ and every set belonging to $\mathcal{S}$ is equal to some T_i. This is clearly just equal to $\mathcal{S}$ itself so by the above lemma, A belongs to $\mathcal{S}$ if and only if A is definable in the model $\mathcal{R}$. An example is given below:

Example 1.3.8. [3, p. 14] Let $S \subseteq R^m$, $B \in \mathcal{S}_n$ and $f : S \to R^n$ be a function that belongs to $\mathcal{S}$. Then $f^{-1}(B) \in \mathcal{S}_m$.
<u>Solution</u> Van den Dries just gives the expression:

$$x \in f^{-1}(B) \iff \exists y(y \in B \,\&\, (x, y) \in \Gamma(f))$$

as a solution.
<u>Explanation</u> If we take the structure $\mathcal{R} = \langle R, T_i \rangle$ where every set belonging to $\mathcal{S}$ is equal to some T_i. Then, as we are given that B belongs to $\mathcal{S}$ and the graph of f belongs to $\mathcal{S}$, we know that they are definable in $\mathcal{R}$ by Lemma 1.3.7. Thus they can be expressed as formulas $\theta(x_1, ..., x_n)$ & $\psi(x_1, ..., x_{m+n})$, respectively. Now the formula $\exists y_1 ... \exists y_n(\theta(y_1, .., y_n) \wedge \psi(x_1, .., x_m, y_1, ..., y_n))$ defines $f^{-1}(B)$ in $\mathcal{R}$. Hence $f^{-1}(B)$ belongs to $\mathcal{S}$ by the Lemma 1.3.7.

From this point it will just be standard to use the two notions of 'definable' interchangeably, saying that a set A is definable to mean that A belongs to the given $\mathcal{S}$.

Now that we have Lemma 1.3.7 and a better understanding of deciding whether a given set belongs to a structure I will develop on the properties of structures.

The following are exercises taken from [3, p. 15 & p. 23].

Proposition 1.3.9. *(1) Let $\mathcal{S}$ contain binary operations $+ : R^2 \to R$ and $\cdot : R^2 \to R$ with respect to which R is a ring. Then $\mathcal{S}$ contains $\{0\}$ and $\{1\}$ and that if $\mathcal{S}$ contains $A \subseteq R^m$ and the functions $f, g : A \to R$, then it contains $-f$ and $f \cdot g$ from A to R.*

(2) Let $\mathcal{S}$ contain the order relation $\{(x, y) \in R^2 : x < y\}$ and give the sets in $\mathcal{S}$ the topology such that the intervals form a base in R and cartesian products of intervals form a base in R^m (see Remark 2.1.1 for more information on the topology), then the topological closure cl(A) of a definable set $A \subseteq R^m$ belongs to $\mathcal{S}$.

(3) Let $R = \mathbb{R}$ and $\mathcal{S}$ contain addition and multiplication, then $\mathcal{S}$ contains the order relation and each singleton $\{q\}$ such that q is a rational number.

(4)Let $R = \mathbb{R}$ and $\mathcal{S}$ contain addition, multiplication, ordering, 0 and 1 with respect to an ordered field, then each function $f : \mathbb{R}^m \to \mathbb{R}$ defined by a polynomial $f(X_1, ..., X_m) \in \mathbb{R}[X_1, ..., X_m]$ belongs to $\mathcal{S}$.

Proof. (1) We can see that 0 is defined by $x = 0 \iff$ for all $y \in R$, $x + y = y$. Similarly 1 is defined by $x = 1 \iff$ for all $y \in R$ $x \cdot y = y$. $-f$ is defined by $(x, y) \in \Gamma(-f) \iff (x, -y) \in \Gamma(f)$ (note that we are supposing that '$-$' is definable but this is clearly the case). $f \cdot g$ is defined by $(x, y) \in \Gamma(f \cdot g) \iff$ there exists $z, z' \in \Gamma(f)$ such that $f(x) = z$ & $g(x) = z'$ &$z \cdot z' = y$.

(2) $(x_1, .., x_m) \in \mathrm{cl}(A) \iff$ for all $a_1, ..., a_m, b_1, ..., b_m \in R$ if $a_1 < x_1 < b_1, ..., a_m < x_m < b_m$ then there is $(c_1, ..., c_m) \in A$ such that $a_1 < c_1 < b_1, ..., a_m < c_m < b_m$.

(3) First see that we can define the set of positive elements of R by r is positive $\iff$ there exists some $k \in R$ such that $k \cdot k = r$. We then define the order relation: $x < y \iff$ for some positive r, $x + r = y$. Let $q = a/b$ be some rational number. Then due to (1), 1 is definable and so then are a and b for example by $x = a \iff x = 1 + ... + 1$ (a times). Then, $x = q \iff b \cdot x = a$.

(4) Let $f(X_1, ..., X_m) = \sum_{i=0}^{d} a_i X_1^{\delta_1(i)} \cdot ... \cdot X_m^{\delta_m(i)}$ for some $d \in \mathbb{N}$, $\delta_j(i) \in \mathbb{N}$ and $a_i \in \mathbb{Z}$. Again, $\mathcal{S}$ contains 1 and addition so it also contains all the integers. Then $f(x_1, ..., x_m) = y \iff y = a_0 x_1^{\delta_1(0)} \cdot ... \cdot x_m^{\delta_m(0)} + ... + a_d x_1^{\delta_1(d)} \cdot ... \cdot x_m^{\delta_m(d)}$ defines f. This actually shows that each *semialgebraic set* in $\mathbb{R}^m$ belongs to $\mathcal{S}$. I will discuss these later. (They are defined in 1.4.7).

□

1.4. o-Minimal Structures

We have seen already that structures have some useful properties, however we can add the conditions of o-minimality which restrict us to working with a smaller class of objects but turn out to give some very nice properties. Importantly, there are some well-studied and often-used structures which are contained in the class of o-minimal structures. Now that we have an understanding of structures we can look at what it means for a given structure to be o-minimal, the core property of this investigation. First, however, it will be necessary to define some terms.

Definition 1.4.1. *[3, p. 17] Given a set R that is linearly ordered by $<$ (we say that $(R, <)$ is a linearly ordered set) we say that R is **dense** if for all $a, b \in R$ with $a < b$ there is $c \in R$ with $a < c < b$. We say that $(R, <)$ is a **dense linearly ordered set**.*

From now on we will let $(R, <)$ be a dense linearly ordered nonempty set without endpoints.

Remark 1.4.2. [3, p. 17] Given $(R, <)$, R does not itself contain endpoints but we write $+\infty$, $-\infty$ to denote endpoints such that for all $r \in R$ $-\infty < r < +\infty$. Also, we write R_∞ to denote the set $R \cup \{-\infty, +\infty\}$.

Definition 1.4.3. *[3, p. 17] An **interval** is a nonempty set $(a, b) := \{x \in R : a < x < b\}$ with $-\infty \leq a < b \leq +\infty$.*

Definition 1.4.4. *[3, p. 17] Let $(R, <)$ be a dense linearly ordered non empty set without endpoints. An **o-minimal structure** on $(R, <)$ is by definition a structure $\mathcal{S}$ on R such that*

(O1) $\{(x, y) \in R^2 : x < y\} \in \mathcal{S}_2$;

(O2) the sets in $\mathcal{S}_1$ are exactly the finite unions of intervals and points.

We say that $(R, <, \mathcal{S})$ is an o-minimal structure.

In essence, O1 tells us that the order relation is definable and O2 that all the finite unions of intervals and points are definable and that these are the only definable subsets of R, note that these are exactly the subsets of R definable using only the order relation and constants.

From now on we will assume that $(R, <, \mathcal{S})$ is an o-minimal structure.

Van den Dries now gives the following lemma, showing simple properties that will be useful later which follow from o-minimality. Van den Dries does not provide a proof so I will provide a quick proof.

Lemma 1.4.5. *[3, p. 18] Let $A \subseteq R$ be definable. We let inf(A) and sup(A) denote the usual infinum and supremum of $A \subseteq R$. Then:*

(i) inf(A) and sup(A) exist in R_∞,

(ii) the boundary,

$$bd(A) := \{x \in R : \text{ each interval containing } x \text{ intersects both A and } R - A\},$$

is finite, and if $a_1 < ... < a_k$ are the points of bd(A) in order, then each interval $(a_i, a_{i-1)}$, where $a_0 = -\infty$ and $a_{k+1} = +\infty$, is either part of A or disjoint from A.

Proof. (i) As A is definable then from the definition of o-minimality, A is a finite union of intervals and points, $A = \bigcup_{i \in I} B_i$. I is a finite indexing set, so there is some B_i containing a point $r \in R$ that is greater than or equal to any point in any of the other sets B_j. If B_i is a singleton set then $r = sup(A)$. If $B_i = (a, b)$ for some $a, b \in R$, then $b = sup(A)$. This concludes the proof for sup(A). The proof for inf(A) is similar.

(ii) Let $A = \bigcup_{i \in I} B_i$, where the B_i are intervals or points in R and I is a finite indexing set (this comes from o-minimality). We can also assume that each B_i is disjoint. Then for each j, B_j is a singleton set or an interval. For each B_j that is a singleton set, we get (at most) one point in the boundary of A & for each B_j that is an interval we get (at most) two points in the boundary of A and these are the only boundary points. As I is finite, so must be bd(A).

For the second part, note that if (a_i, a_{i+1}) is not a subset of A and is not disjoint from A then there must be a boundary point of A in between a_i and a_{i+1} which is a contradiction.

□

So far we have described a set-theoretic definition of o-minimality. How does this transfer to a model-theoretic notion?

Definition 1.4.6. *[3, p. 23] A model-theoretic structure $\mathcal{R} = (R, <, ...)$, where $<$ is a dense linear order without endpoints on R, is called **o-minimal** if Def$(\mathcal{R}_R)$ is an o-minimal structure on $(R, <)$, in other words, every set $S \subseteq R$ that is definable in $\mathcal{R}$ using constants is a union of finitely many intervals and points.*

There is an important distinction that should be noted. In speaking of an o-minimal structure *on* $(R, <)$, we mean in the sense of Definition 1.4.4. However, we will denote o-minimal structures in the model-theoretic sense by $\langle R, <, ... \rangle$; in particular we may look at the model-theoretic structure $\langle R, < \rangle$ and this should not be confused with the notion of an o-minimal structure on $(R, <)$, this should be clear from the context.

1.4.1. Examples of o-Minimal Structures

Example 1.4.7. (1) The model-theoretic structure $\langle R, < \rangle$ is o-minimal where $(R, <)$ is a dense linearly ordered set without endpoints [3, p. 24]. It is intuitively very clear why this is the case, using $<$ and parameters to define subsets of R we can only define unions of finitely many intervals and points.

(2) The semialgebraic sets on $(\mathbb{R}, <)$ [3, p. 37]. The semi-algebraic sets on $(\mathbb{R}, <)$ are defined as follows: Let $k, l, n \in \mathbb{N}$ and $f_1, ..., f_k, g_1, ..., g_l \in \mathbb{R}[X_1, ..., X_n]$. Define a set, $X := \{x \in \mathbb{R}^n : f_1(x) = ... = f_k(x) = 0, g_1(x) > 0, ..., g_l(x) > 0\}$. A semialgebraic set is then a finite union of sets defined by polynomials in this way [3, p. 1]. Semialgebraic sets will be discussed further in Chapter 5.

(3) The ordered field of real numbers, $\mathcal{R} = \langle \mathbb{R}, <, 0, 1, +, \cdot \rangle$. This actually follows from the fact that the sets definable in $\mathcal{R}$ using constants are exactly the semialgebraic sets on $(\mathbb{R}, <)$ [3, p. 37].

(4) The exponential field of real numbers, $\mathcal{R} = \langle \mathbb{R}, <, 0, 1, +, \cdot, \exp \rangle$ - this will be discussed later in Chapter 4.

(5) The exponential field of real numbers with restricted analytic functions is also o-minimal. This was proven in [7]. Given an analytic function in many variables, $f : \mathbb{R}^m \to \mathbb{R}$, which converges in $[-1, 1]^m$. Then $\bar{f} : \mathbb{R}^m \to \mathbb{R}$ defined by

$$\bar{f}(x) = \begin{cases} f(x) & \text{for } x \in [-1,1]^m \\ 0 & \text{for } x \notin [-1,1]^m \end{cases}$$

is a restricted analytic function.

2. Monotonicity and Cell Decomposition

2.1. Introduction

Now that we have looked at the basic notions of o-minimality we can move onto two of the most fundamental results in the field. Firstly, the monotonicity theorem. This says that any definable function on an interval in an o-minimal structure acts uniformly except for at a finite number of points. This is a very useful result and will be used in later proofs. Given some definable function it allows us to split into cases where the function is continuous and strictly monotone or constant.

Secondly, the cell decomposition theorem. We define inductively the notion of a cell in R^m, this is either a point, interval, the graph of some definable function on a given cell or the space between the graphs of two distinct definable functions on a cell. Next we define the notion of a decomposition of R^m into cells, this is a finite partition of R^m into cells. Finally, the theorem tells us that given any finite collection of definable sets in R^m there is a decomposition of R^m into cells which partitions each of the given sets.

In later chapters we will make frequent use of both these theorems, highlighting how important they are.

Remark 2.1.1. Throughout this chapter fix an o-minimal structure $(R, <, \mathcal{S})$ and as usual we say that a set $A \subseteq R^m$ is definable to mean that A belongs to $\mathcal{S}$.

Also, throughout the rest of the report we equip R with the interval topology - the open sets of R are thus the intervals and any union of intervals. We equip R^m with the induced product topology - the cartesian products of intervals in R and any union of such products are the open sets. So when we speak of properties such as continuity we mean in the topological sense (See Aside 1). Also note that R^m is a hausdorff space with this topology (this is easy to see given that the ordering, $<$, on R is dense) [3, p. 17].

2.2. Monotonicity

We begin with the statement of the monotonicity theorem.

Theorem 2.2.1. *[3, p. 43] Monotonicity Theorem. Let $f : (a, b) \to R$ be a definable function on the interval (a, b). Then there are points $a_1, ..., a_k$ in (a, b) such that on each subinterval (a_j, a_{j+1}), with $a_0 = a$, $a_{k+1} = b$, the function is either constant, or strictly monotone and continuous.*

This theorem means that a definable function in one variable can be cut into pieces such that on each piece the function is well-behaved.

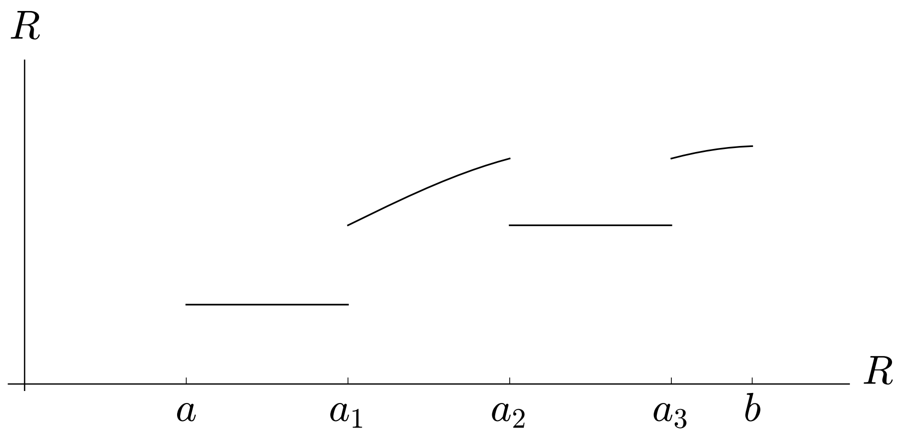

Figure 2.1.: Example of a function on R which can be cut up 'nicely'

In the above figure 2.1 the function shown is not continuous or strictly monotone on the interval (a, b), however taking $a_0 = a, a_4 = b$ the function is then either constant, or strictly monotone and continuous on the intervals (a_i, a_{i+1}) with $0 \leq i \leq 3$.

This is not trivial however; consider a function such as $f(x)$ =sin(x) on the interval $\mathbb{R} = (+\infty, -\infty)$. This cannot be cut into pieces in such a way as above as we would need an infinite number of subintervals. This shows, in fact, that $f(x)$ =sin(x) on the interval $(+\infty, -\infty)$ is not definable for any o-minimal structure. For an even better example consider the Weierstrass function on an interval of the real line which is continuous everywhere but not strictly monotone or constant on any subinterval.

The theorem was first proved in [1] by Pillay and Steinhorn.

As this is an important theorem I will give the proof that is given in [3], in places giving a little more detail. We first begin by proving the following lemma:

Lemma 2.2.2. *[3, p. 43] Let $f : I \to R$ be a definable function on an interval I.*

1) There is a subinterval of I on which f is constant or injective.

2) If f is injective, then f is strictly monotone on a subinterval of I.

3) If f is strictly monotone, then f is continuous on a subinterval of I.

I will first consider the proof of this lemma and then explain how the monotonicity theorem follows from them.

Van den Dries does not mention o-minimality explicity in the proof he gives of this lemma, so I will give the proof as given by van den Dries [3, p. 44] but will add in some explanation to make clear why the o-minimality condition is important.

Proof. 1 Suppose that for some $y \in R$ the preimage, $f^{-1}(y)$, is infinite. Then this preimage contains a subinterval of I. Why is this the case? Well, we have seen in an earlier example (1.3.8) that the preimage of a definable set under a definable function is itself definable. Hence, $f^{-1}(y)$ is definable. So, we have an infinite, definable set $f^{-1}(y) \subseteq R$. By o-minimality, $f^{-1}(y)$ is a finite union of intervals and points of R. In order for $f^{-1}(y)$ to be infinite it must then contain at least one interval. Hence, $f^{-1}(y)$ contains a subinterval of I. In this case, clearly f takes a constant value, y, on this subinterval of I.

So, we may assume that $f^{-1}(y)$ is finite for all $y \in R$. Then, as I is infinite, it must be the case that $f(I)$ is infinite.

$f(I)$ is definable [this can be seen by the formula: $y \in f(I) \iff \exists x(x \in I \ \& \ (x, y) \in \Gamma(f))$] and $f(I)$ is infinite. So again, as above, by o-minimality, $f(I)$ must contain an interval, J.

Define a function $g : J \to I$ by

$$g(y) := min\{x \in I : f(x) = y\}$$

Clearly g is injective. This also implies that $g(J)$ is infinite (as J is infinite). We can also see that $g(J)$ is definable. Again this implies, by o-minimality, that $g(J)$ contains a subinterval of I.

Then f is injective on this subinterval of I, as suppose not, then for some $c, d \in g(J)$ with $c < d$, $f(c) = f(d)$. Then, $g(f(d)) = g(f(c)) \leq c \Rightarrow d \notin g(J)$. Contradiction.

2 The proof here is rather lengthy and not very enlightening so I will omit this.

3 Assume that f is strictly increasing (the proof when f is strictly decreasing is essentially the same). Then $f^{-1}(y)$ is finite for all $y \in R$. Hence, as in (1), $f(I)$ is infinite and so contains an interval, $J \subseteq f(I)$.

Let $r, s \in J$ be such that $r < s$ and let c, d be their respective preimages i.e. $f(c) = r, f(d) = s, c < d$. Now, $f : (c, d) \to (r, s)$ is an order preserving bijection (as f is strictly increasing). Take an open set, $V = (r', s') \subseteq (r, s)$. Then clearly $f^{-1}(V) = (c', d')$ for some c', d' with $c \leq c' < d' \leq d$. Hence, f is continuous on (c, d).

□

Now that we have the above lemma, we can move on to prove the monotonicity theorem.

Proof of Monotonicity Theorem. [3, p. 44] Let

$$X = \{x \in (a, b) : \text{on some interval containing } x \text{ the function } f \text{ is either constant, or strictly monotone and continuous}\}$$

Suppose that $(a, b) - X$ is infinite. Then, by o-minimality, $(a, b) - X$ contains an interval, I. Consider $f : I \to R$. As I is definable and f is definable we can apply Lemma 2.2.2 1) If there is a subinterval of I on which f is constant then $I \cap X \neq \emptyset$ which is a contradiction. So there must be a subinterval, J of I where f is injective. Then, by applying (2) to J we get that f is strictly monotone on a subinterval of J. Applying (3) we get that f is strictly monotone and continuous on a subinterval of I. Hence, $I \cap X \neq \emptyset$ which is a contradiction.

Hence, $(a, b) - X$ is finite. So there are a finite number of points in (a, b) at which, if any interval contains them then f is neither constant nor strictly monotone and continuous on that interval. Hence X is in fact a finite union of intervals. We consider an arbitary one of these intervals, (c, d) say.

Now, at every point $x \in (c, d)$, f is continuous on an interval around x. Hence, f is continuous on (c, d).

We can now split into three cases:

1 for all $x \in (c, d)$, f is constant on a neighbourhood of x

2 for all $x \in (c, d)$, f is strictly increasing on a neighbourhood of x

3 for all $x \in (c, d)$, f is strictly decreasing on a neighbourhood of x

Case 1: Let $x' \in (c, d)$. Set

$$s := sup\{x \in (c, d) : x' < x < d, f \text{ is constant on } [x', x)\}$$

Suppose for a contradiction that $s < d$. Then $s \in (c, d)$, so f is constant on some neighbourhood of s, say $(s - i, s + i)$ where we take i so that $(s - i, s + i) \subseteq (c, d)$, then $s + i > s$ and $s + i \in sup\{x \in (c, d) : x < x < d, f \text{ is constant on } [x', x)\}$ which is a contradiction. Hence, $s = d$.

This proves that f is constant on $[x', d)$. We can prove in the same way that f is constant on $(c, x']$. Hence, f is constant on (c, d).

Case 2: The argument here is very similar to case one so I will omit this, we show that f is strictly increasing on (c, d) instead of constant.

Case 3: It is shown that f is strictly decreasing on (c, d), again I will omit this argument as it is similar to case 1.

So, we have split (a, b) into a finite number of intervals on which f is either constant, or strictly monotone and continuous. This concludes the proof. □

The following lemma is considered a basic result by van den Dries and will be useful later on. The proof introduces some new concepts and is quite lengthy so I will not include it. I will, however give a proof for a result combining the following lemma with the monotonicity theorem.

It will first be necessary to define the term *fiber*.

Definition 2.2.3. *[3, p. 59] Let $A \subseteq R^{m+n}$ be definable. Then for each $x \in R^m$ the* ***fiber****, A_x, is defined to be the set $\{y \in R^n : (x, y) \in A\} \subseteq R^n$.*

For a two dimensional representation of this see Figure 2.2.

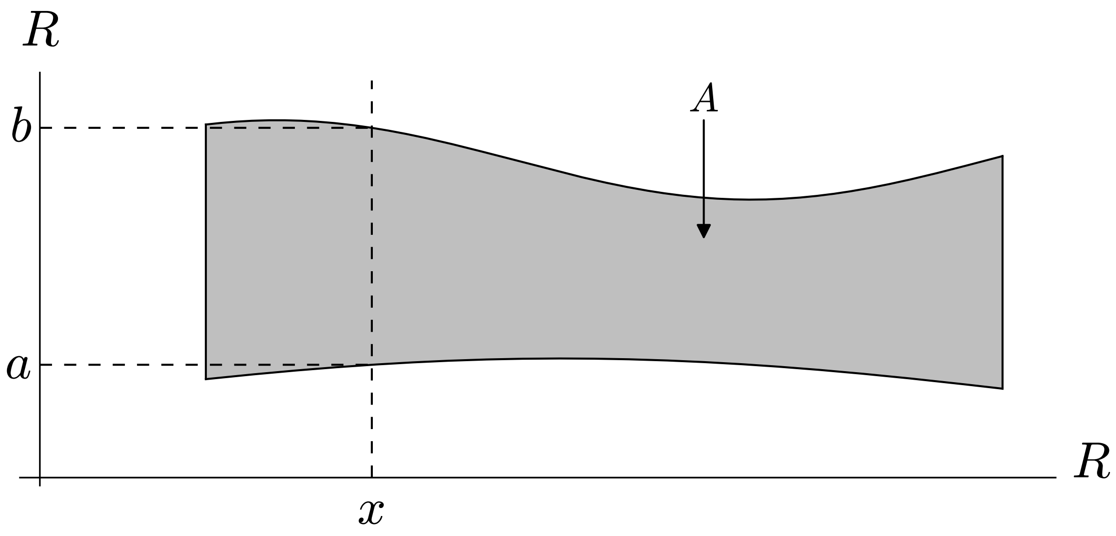

Figure 2.2.: Representation of the fiber A_x

We can see that in Figure 2.2 the fiber A_x is just the interval (a, b) and the boundary of A_x, bd(A_x), is the set $\{a, b\}$.

Example 2.2.4. Let $f : R^m \to R$ be some function and let

$$A = \Gamma(f) = \{(x, f(x)) : x \in R^m\}.$$

Then the fiber A_x is just the set containing the point $f(x)$.

Remark 2.2.5. Note that the notation for indexing sets and fibers as above is the same. It should be clear which is meant from the context and generally subscripts x or y will be used for fibers and k or λ used for indexing sets.

Given a definable set $A \subseteq R^{m+n}$ and $x \in R^m$, the fiber A_x is clearly definable, to see this note that $y \in A_x \iff (x, y) \in A$. This is a fact that will be used frequently throughout.

Lemma 2.2.6. *[3, p. 46] Finiteness Lemma. Let $A \subseteq R^2$ be definable and suppose that for each $x \in R$ the fiber $A_x := \{y \in R : (x, y) \in A\}$ is finite. Then there is $N \in \mathbb{N}$ such that $|A_x| \leq N$ for all $x \in R$.*

If it is the case that for each $x \in R$ there is only a finite number of points, $y \in R$ such that $(x, y) \in A$, then by the above lemma there is an upper bound, N, such that given any $x \in R$ there is no more than N points $y \in R$ such that $(x, y) \in A$.

Proof. Omitted. □

The following result is given by van den Dries but a proof is not given. I will give a proof.

Lemma 2.2.7 (Combining Monotonicity and Finiteness)**.** *[3, p. 49] Let $A \subseteq R^2$ be definable such that A_x is finite for each $x \in R$. Then there are points $a_1 < ... < a_k$ in R such that the intersection of A with each vertical strip $(a_i, a_{i+1}) \times R$ has the form $\Gamma(f_{i1}) \cup ... \cup \Gamma(f_{in(i)})$ for some definable continuous functions $f_{ij} : (a_i, a_{i+1}) \to R$ with $f_{i1}(x) < ... < f_{in(i)}(x)$ for all $x \in (a_i, a_{i+1})$. (We set $a_0 = -\infty, a_{k+1} = +\infty$).*

Proof. First, we need to prove that we can split R into a finite number of intervals (b_i, b_{i+1}) such that for every i and each $x, y \in (b_i, b_{i+1})$, $|A_x| = |A_y|$.

Let

$$X = \{x \in R : \text{for some subinterval } (c, d) \text{ of } R \text{ containing } x, \text{ for all } y \in (c, d)|A_x| = |A_y|\}.$$

Assume for a contradiction that $R - X$ is infinite. Note that X is clearly definable and thus so is $R - X$. Hence, $R - X$ contains an interval by o-minimality. Let's call this interval (r, s).

By the Finiteness Lemma we know that there is an $N \in \mathbb{N}$ such that $|A_x| \leq N$ for all $x \in R$. In particular, $|A_x| \leq N$ for all $x \in (r, s)$. As (r, s) is infinite and the fibers are bounded by N there must be an $n \in \mathbb{N}$ such that there are infinitely many $y \in (r, s)$ with $|A_y| = n$.

Let Y be this subset of (r, s) i.e. for all $y \in Y, |A_y| = n$. Then Y is infinite. Y is clearly also definable. Hence Y contains an interval by o-minimality. This is a contradiction as Y is a subset of $R - X$. Hence $R - X$ is finite.

Let $R - X = \{b_1, ..., b_k\}$. Similar to the arguments about the three cases in the monotonicity proof, it is clear that for each interval (b_i, b_{i+1}) and every $x, y \in (b_i, b_{i+1})$, $|A_x| = |A_y|$. Now, for each i, set $n(i) = |A_x|$ where $x \in (b_i, b_{i+1})$.

We can now define functions on each interval (b_i, b_{i+1}) as follows:

$$f_{i1}(x) = minA_x$$

$$f_{i2}(x) = min(A_x - \{f_{i1}(x)\})$$

$$...$$

$$f_{in(i)}(x) = min(A_x - (\{f_{in(i)-1}(x)\} \cup ... \cup \{f_{i1}(x)\})$$

Then for each i the functions are well defined and satisfy the conditions, $f_{i1}(x) < ... < f_{in(i)}(x)$ and $A \cap ((b_i, b_{i+1}) \times R) = \Gamma(f_{i1}) \cup ... \cup \Gamma(f_{in(i)})$.

However, it may not be the case that every function, f_{ij}, is continuous on the subinterval (b_i, b_{i+1}). Applying the monotonicity theorem to each f_{ij} we further split up the intervals (b_i, b_{i+1}) into subintervals (a_i, a_{i+1}) on which the functions are continuous and hence satisfy the conditions as required. □

The following example is given as an exercise by van den Dries, it shows that the points $a_1, ..., a_k$ in the statement of the monotonicity theorem are definable. It will also be used later to prove a strengthened version of cell decomposition.

Proposition 2.2.8. *[3, p. 49] Suppose the function $f : (a, b) \to R$ on the interval (a, b) is definable. Show that there exist elements $a_1, ..., a_k$ with the property of the monotonicity theorem such that $a_1, ..., a_k$ are definable in the model-theoretic structure $\langle R, <, \Gamma(f)\rangle$.*

Proof. Split the interval (a, b) using the monotonicity theorem to obtain points $a_1, ..., a_k$ such that f is continuous and strictly monotone or constant on the intervals (a_i, a_{i+1}) for $0 \leq i \leq k + 1$ where $a_0 = a$ and $a_{k+1} = b$. We may suppose that splitting the interval in this way does so with the least number of points necessary.

First note that a_0 is definable in$\langle R, <, \Gamma(f)\rangle$ as it is the infimum of the definable set $\pi(\Gamma(f))$.

Then $x = a_1 \iff a_0 < x$ and f is either constant or strictly monotone and continuous on (a_0, x) and for all $t > x$, f is neither constant nor strictly monotone and continuous on (a_0, t).

This is a slightly informal solution for defining a_1 in $\langle R, <, \Gamma(f)\rangle$, though it should be clear that this can be expressed formally.

We can also define $a_2, ..., a_k$ in $\langle R, <, \Gamma(f)\rangle$ in a similar way, for example $x = a_2$ if $x > a_1$ and f is either constant or strictly monotone and continuous on (a_1, a_2) and f is neither constant nor strictly monotone and continuous on any interval (a_1, t) with $t > a_2$. □

2.3. Cell Decomposition

2.3.1. Cells

In order to get to the main theorem of this chapter, it will be necessary to introduce the idea of a cell. Due to the cell decomposition theorem cells have an important place in work on o-minimality so I will also give a result regarding the topological nature of cells. They are also interesting as each cell is homeomorphic to R^k for some k.

We begin by giving some notation.

Let $X \subseteq R^m$ be a definable set. We define

$$C(X) := \{f : X \to R : f \text{ is definable and continuous}\}$$

i.e. the set containing all definable and continuous functions on X. We define

$$C_\infty(X) := C(X) \cup \{-\infty, +\infty\}$$

i.e. $C(X)$ along with the constant functions mapping any element of X to $-\infty$ or $+\infty$ [3, p. 49].

We write $f < g$ to mean that for all $x \in X$, $f(x) < g(x)$. Let $f, g \in C_\infty$ and $f < g$, we define

$$(f, g)_X := \{(x, r) \in X \times R : f(x) < r < g(x)\}.$$

As noted by van den Dries this is a definable set [3, p. 49].

Below we give an inductive definition of cells.

Definition 2.3.1. *[3, p. 50] Let $(i_1, ..., i_m)$ be a sequence of zeros and ones of length m. An $(i_1, ..., i_m)$-**cell** is a definable subset of R^m obtained by induction on m as follows:*

(i) a (0)-cell is a one element set $\{r\} \subseteq R$, a (1)-cell is an interval $(a, b) \subseteq R$;

(ii) suppose $(i_1, ..., i_m)$-cells are already defined; then an $(i_1, ..., i_m, 0)$-cell is the graph, $\Gamma(f)$ of a function $f \in C(X)$, where X is an $(i_1, ..., i_m)$-cell; a $(i_1, ..., i_m, 1)$-cell is a set $(f, g)_X$ where X is an $(i_1, ..., i_m)$-cell and $f, g \in C_\infty$, $f < g$

For a graphical interpretation of some cells in R and R^2, see Figure 2.3.

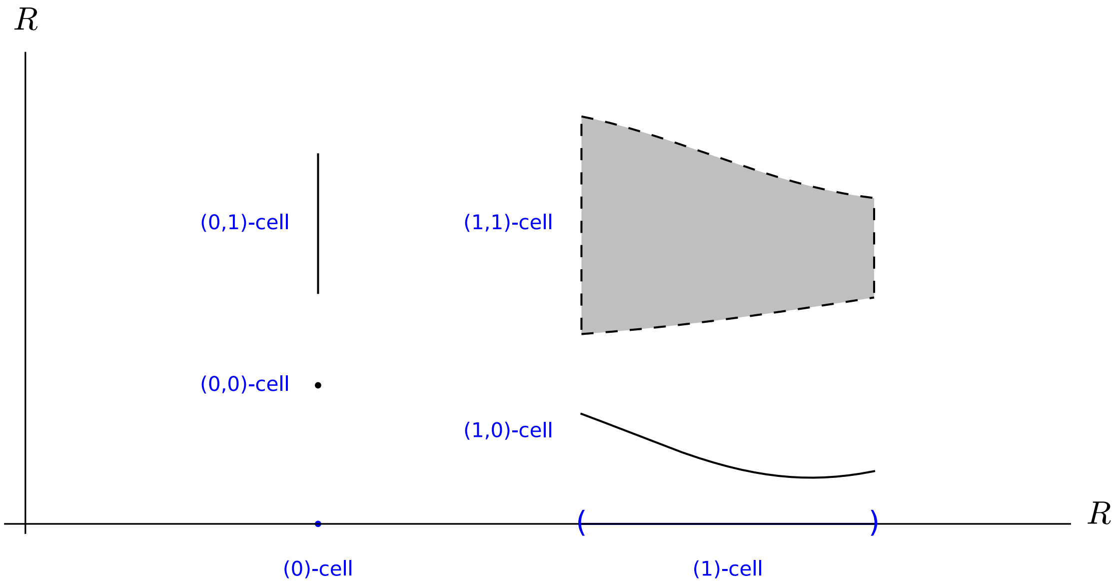

Figure 2.3.: Cells in R^2

Remark 2.3.2. [3, p. 50] We call the $(1,...,1)$-cells the open cells as they are the cells which are open in the ambient space. Also, we can see that by 'stretching' the $(1,1)$-cell we can obtain the entire space R^2, similarly with the $(0,1)$ and $(1,0)$ cells we can obtain R. In topological terms, these cells are homeomorphic to the respective spaces. This can be extended to any $(i_1,..,i_m)$-cell with it being homeomorphic to R^n where $n = \sum_{j=1}^{m} i(j)$. This is the content of the following definition.

We define a function which will be used in the proof of the cell decomposition theorem. It takes any cell and performs a coordinate projection which removes the co-ordinates corresponding to zeroes.

Definition 2.3.3. *[3, p. 51] Let $i = (i_1,...,i_m)$. We define the function $p_i : R^m \to R^k$ as follows: let $\lambda(1) < ... < \lambda(k)$ be the indices $\lambda \in \{1,..,m\}$ for which $i_\lambda = 1$, so that $k = i_1 + ... + i_m$; then*

$$p_i(x_1,...,x_m) := (x_{\lambda(1)},...,x_{\lambda(k)}).$$

For an i-cell A we denote $p_i(A)$ by $p(A)$ and $p_i|A : A \to p(A)$ by p_A.

Van den Dries notes that this map is a homeomorphism, taking each cell A in R^m to an open cell in R^k. What will be particularly important later is that this map is bijective, so I will provide a proof of this. If we consider Figure 2.3; we can see that the $(0,1)$-cell can be bijectively mapped onto the vertical axes and the $(1,0)$-cell can be mapped bijectively onto the horizontal axes. Below I will make this more formal.

Lemma 2.3.4. *Let $p_i : R^m \to R^k$ be the function as defined above and let $A \subseteq R^m$ be an i-cell. Then $p_i|A : A \to p(A)$ is a bijection.*

Proof. We prove by induction on m.

Base Case Let $m = 0$ (by convention we consider the one-point space R^0 as an open $()$-cell - this being the only cell in R^0. Note that graphs of functions on this cell give all the (0)-cells and the intervals between continuous functions on this cell gives all the (1)-cells so it is appropriate to start the induction here). Let $i = ()$. If A is an i-cell, i.e. $A = R^0$ then $p_i|A : A \to p(A)$ is just the identity map and so is bijective.

Induction Hypothesis Let the result hold for m. Consider $m+1$, so $i = (i_1, ..., i_{m+1})$. There are two cases to consider, $i = (i_1, ..., i_m, 0)$ and $i = (i_1, ..., i_m, 1)$.

Let $A \subseteq R^{m+1}$ be an $(i_1, ..., i_m, 0)$-cell. Then by definition there is some $(i_1, ..., i_m)$-cell, X, and function $f \in C(X)$ such that $A = \Gamma(f)$. Then the projection map, $\pi : A \to X$, on the first m coordinates is bijecitive (this is easy to check). By the inductive hypothesis p_X is bijective, hence the function $p_X \circ \pi$ on A is bijective and it is clear that this is just $p_i|A : A \to p(A)$.

Let $A \subseteq R^{m+1}$ be an $(i_1, ..., i_m, 1)$-cell. Let $X = \pi(A)$. Consider the function $h : X \times R \to p(X) \times R$ defined by $(x, y) \mapsto (p_X(x), y)$. As p_X is bijective then so is $h|A$ and it is clear that $h|A = p_i|A$. □

Before we introduce the following proposition we introduce a topological notion of connectedness as applied to definable sets.

Definition 2.3.5. *[3, p. 19] A set $X \subseteq R^m$ is called* ***definably connected*** *if X is definable and X is not the union of two disjoint nonempty definable open subsets of X.*

This is essentially the same as the usual topological notion of connectedness but with the condition of being definable added.

We now prove some basic results about definable connectedness as this will be useful later. The following lemma is given as an exercise by van den Dries but the proof is left as an exercise.

Lemma 2.3.6. *[3, p. 19] The following are definably connected subsets of R:*

(i) The empty set,

(ii) The intervals,

(iii) The sets $[a, b)$ with $-\infty < a < b \leq +\infty$ and $(a, b]$ with $-\infty \leq a < b < +\infty$,

(iv) The sets $[a, b]$ with $-\infty < a \leq b < +\infty$.

Proof. (i) Clearly the empty set is definably connected as it cannot be the union of two non-empty sets.

(ii) Let $X = (a, b)$ for some $a, b \in R$. Then $|\text{bd}(X)| = 2$. Suppose that X is the union of two disjoint nonempty definable open subsets, $X = (c, d) \cup (x, y)$. Then $|\text{bd}(X)| = 4$. Contradiction. Hence X is definably connected. (Note: the case where $a, b \in R_\infty$ is not included here but is easy to prove).

(iii) Let $X = [a, b)$ with $-\infty < a < b \leq +\infty$ and suppose that X is the union of two disjoint nonempty definable open subsets, $X = (c, d) \cup (x, y)$. Then $a \in X \Rightarrow$ (wlog) $a \in (c, d) \Rightarrow$ there is some $a' \in R$ with $c < a' < a$ and $a' \in X$. Contradiction. The case for $(a, b]$ is similar.

(iv) The proof of this case is similar to (iii) so I will omit this.

□

We can now look at a result of definable connectedness applied to cells, telling us that each cell is definably connected. Van den dries gives a brief proof of this but I will give some more detail.

Proposition 2.3.7. *[3, p. 51] Each cell is definably connected.*

Proof. The proof is by induction. Let $X \subseteq R$ be a cell. Then X is an interval or point, hence X is definably connected by 2.3.6 (ii) or (iv), respectively.

Note that if C is a cell in R^{m+1} then $\pi(C)$ is a cell in R^m where π is the projection map on the first m coordinates. Suppose that all cells in R^m are definably connected. Let X be a cell in R^{m+1}. Suppose for a contradiction that X is a union of two disjoint nonempty definable open subsets of X, i.e.

$$X = A \cup B = (a_1, b_1) \times ... \times (a_{m+1}, b_{m+1}) \cup (a_1', b_1') \times ... \times (a_{m+1}', b_{m+1}').$$

Hence $\pi(X) = \pi(A) \cup \pi(B) = (a_1, b_1) \times ... \times (a_m, b_m) \cup (a_1', b_1') \times ... \times (a_m', b_m')$. If $\pi(A)$ and $\pi(B)$ are disjoint then $\pi(X)$ is not definably connected. Contradiction.

Suppose then that $\pi(A)$ and $\pi(B)$ are not disjoint. Then there is $x \in \pi(A) \cap \pi(B)$ where $\pi^{-1}(x) \cap X$ is not definably connected as

$$\pi^{-1}(x) \cap X = \{x\} \times (a_{m+1}, b_{m+1}) \cup \{x\} \times (a_{m+1}', b_{m+1}')$$

which are disjoint definable open subsets of $\pi^{-1}(x) \cap X$. This is a contradiction as for every cell $C \subseteq R^{m+1}$ and $x \in R^m$ the set $\pi^{-1}(x) \cap C$ is definably connected.

To see this take a cell $X \subseteq R^{m+1}$ and $x \in R^m$. Consider

$$\pi^{-1}(x) \cap X = \{(x, y) : y \in R\}$$

and suppose $\pi^{-1}(x) \cap X = A \cup B$ for some A, B nonempty definable open subsets of $\pi^{-1}(x) \cap X$. If X is a $(i_1, ..., i_m, 0)$-cell then $\pi^{-1}(x) \cap X = \{(x, f(x))\}$ (for the appropriate function f) hence $A = B = \{(x, f(x))\}$ and so $\pi^{-1}(x) \cap X$ is definably connected. If X is a $(i_1, ..., i_m, 1)$-cell then $\pi^{-1}(x) \cap X = \{(x, y) \in X : c < y < d\}$ for some $c, d \in R_\infty$. Then A and B are not disjoint. Hence $\pi^{-1}(x) \cap X$ is definably connected. □

2.3.2. Decomposition

Definition 2.3.8. *[3, p. 51] A **decomposition** of R^m is a partition of R^m into finitely many cells obtained by induction as follows:*

(i)a decomposition of $R^1 = R$ is a collection

$$\{(-\infty, a_1), (a_1, a_2), ..., (a_k, +\infty), \{a_1\}, ..., \{a_k\}\}$$

where $a_1 < ... < a_k$ are points in R;

(ii) a decomposition of R^{m+1} is a finite partition of R^{m+1} into cells A such that the set of projections $\pi(A)$ is a decomposition of R^m. (Here π is the projection map on the first m coordinates.)

Remark 2.3.9. Van den Dries notes the following [3, p. 52]: Let $D = \{A(1), ..., A(k)\}$ be a decomposition of R^m, $A(i) \neq A(j)$ if $i \neq j$, and let for each $i \in \{1, ..., k\}$ functions $f_{i1} < ... < f_{in(i)}$ in $C(A_i)$ be given. Then

$$D_i := \{(-\infty, f_{i1}), (f_{i1}, f_{i2}), ..., (f_{in(i)}, +\infty), \Gamma(f_{i1}), ..., \Gamma(f_{in(i)})\}$$

is a partition of $A(i) \times R$ and, importantly, $D^* := D_1 \cup ... \cup D_k$ is a decomposition of R^{m+1}. This will be useful in the proof of the following theorem.

Note: Here, van den Dries introduces the notion of a decomposition partitioning a subset of R^m which is necessary for the cell-decomposition theorem. Let D be a decomposition of R^m. Then we say that D **partitions** a set $X \subseteq R^m$ if for each cell $A \in D$, $A \subseteq X$ or $A \cap X = \emptyset$. This is equivalent to saying that X is a disjoint union of cells of D [3, p. 52].

We now give the main theorem of this section.

Theorem 2.3.10. *[3, p. 52] Cell Decomposition Theorem.*

(I_m)Given any definable sets $A_1, ..., A_k \subseteq R^m$ there is a decomposition of R^m partitioning each of $A_1, ..., A_k$.

(II_m)For each definable function $f : A \to R$, $A \subseteq R^m$, there is a decomposition D of R^m partitioning A such that the restriction $f|B : B \to R$ to each cell $B \in D$ with $B \subseteq A$ is continuous.

Again, as this is such a fundamental result to the field I will give the proof as given in [3], adding more detail where necessary. The theorem was first proved in [2] by Knight, Pillay and Steinhorn.

Proof. The proof proceeds via induction on m. We start with (I_1) and (II_1). Van den Dries notes that (I_1) holds by o-minimality and (II_1) holds from the monotonicity theorem. Why is this the case?

(I_1) Well, if $A_1, ..., A_k \subseteq R$ are definable then they are just finite unions of intervals and points (by o-minimality). So by picking $a_1 < ... < a_k$ in R appropriately we get a decomposition of R partitioning each A_i.

(II_1) Let $f : A \to R$, $A \subseteq R$ be a definable function. Then by the monotonicity theorem we split A into intervals (a_i, a_{i+1}) such that f is continuous on each interval. The decomposition of R, $D = \{(-\infty, a_1), ..., (a_k, +\infty)\}$ then gives the required decomposition.

The rest of the proof given by van den Dries is much more involved. I will outline the proof and try to explain some of the geometric intuition behind some of the concepts given.

First it will be necessary to introduce the notion of *uniform finiteness* and give a result about it.

Definition 2.3.11. *[3, p. 53] A set $Y \subseteq R^{m+1}$ is* ***finite over*** *R^m if for each $x \in R^m$ the fiber Y_x is finite. We say that Y is* ***uniformly finite over*** *R^m of there is $N \in \mathbb{N}$ such that $|Y_x| \leq N$ for all $x \in R^m$. (Recall this is the property proved in the finiteness lemma).*

Lemma 2.3.12. *[3, p. 53] Uniform Finiteness Property. Suppose the definable subset $Y \subseteq R^{m+1}$ is finite over R^m. Then Y is uniformly finite over R^m.*

Proof. We see that this is just a generalization of the finiteness lemma to $m > 2$. The proof of this lemma uses the finiteness lemma for the base case and then proves the general case by induction. I will omit this. □

To begin the proof of (I_1) we introduce some notation. Let $A \subseteq R^{m+1}$ be definable, then we set

$$\text{bd}_m(A) := \{(x, r) \in R^{m+1} : r \in bd(A_x)\}.$$

First, note that this set is definable. To get an idea of what this set is first consider Figure 2.4.

We can see that bd$((A_1)_x)$ is the set $\{a, b\}$.

So we attain the entire set bd$_m(A_1)$ by considering each $x \in R^{m+1}$. This is given as blue lines in Figure 2.4.

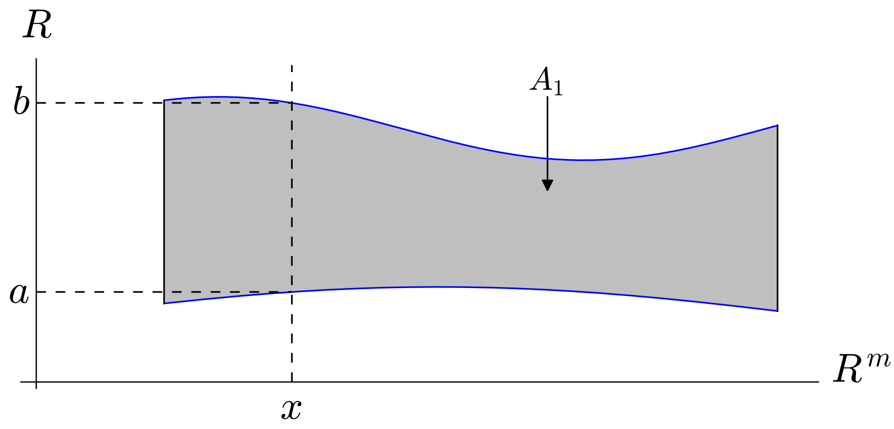

Figure 2.4.: Representation of $\text{bd}_m(A_1)$

Further, take some $x \in R^m$ then the fiber

$$(\text{bd}_m(A))_x = \{r \in R : (x, r) \in \text{bd}_m(A)\} = \text{bd}(A_x)$$

is finite by Lemma 1.4.5. Hence, $\text{bd}_m(A)$ is finite over R^m. Here, A was an arbitrary definable set, so $\text{bd}_m(A)$ is finite over R^m for any definable subset $A \subseteq R^{m+1}$.

(I_{m+1}) We now take definable subsets $A_1, ..., A_k \subseteq R^{m+1}$ and, assuming (I_m) and (II_m), give a decomposition which partitions these subsets.

First, set

$$Y := \text{bd}_m(A_1) \cup ... \cup \text{bd}_m(A_k).$$

Then as each $\text{bd}_m(A_i)$ is definable, so is Y. Also, let $x \in R^m$. Then

$$|Y_x| \leq |(\text{bd}_m(A_1))_x| + ... + |(\text{bd}_m(A_k))_x|$$

which is finite as each $\text{bd}_m(A_i)$ is finite over R^m. Hence, Y is finite over R^m and so Y is uniformly finite over R^m by the uniform finiteness property. Let $M \in \mathbb{N}$ be the bound, i.e. for all $x \in R^m$, $|Y_x| \leq M$.

Now consider $i \in \{0, ..., M\}$ and set $B_i := \{x \in R^m : |Y_x| = i\}$.

We now define functions, $f_{i1}, ..., f_{ii}$ on each B_i by

$$Y_x = \{f_{i1}(x), ..., f_{ii}(x)\}, f_{i1} < ... < f_{ii}.$$

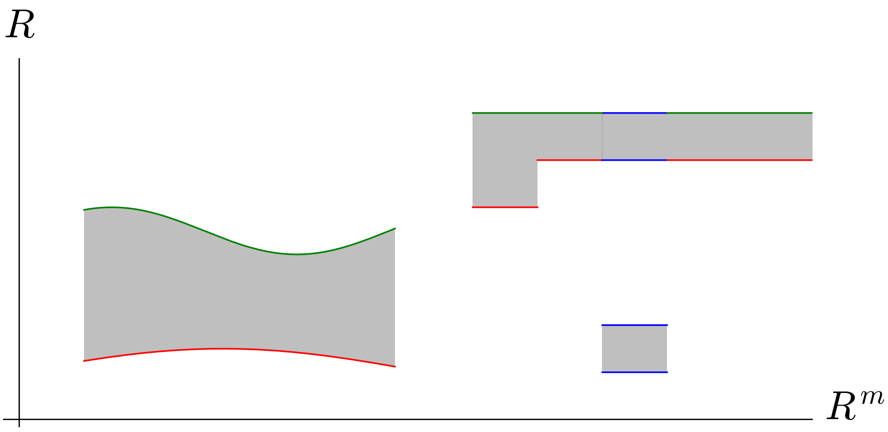

Figure 2.5.: Representation of $f_{i1}, ..., f_{ii}$

Also, for each i we set $f_{i0} := -\infty$ and $f_{ii+1} := +\infty$ as constant functions.

To understand how these functions are defined consider Figure 2.5. Y is the set containing all the green, red and blue lines. The set $B_1 = \emptyset$, B_2 corresponds to the first m coordinates of the green and red lines and B_4 corresponds to the first m coordinates of the blue lines. Moreover, f_{21} corresponds to the red lines and f_{22} corresponds to the green lines.

Now for $\lambda \in \{1, ..., k\}$, $i \in \{0, ..., M\}$ and $1 \leq j \leq i$ we define

$$C_{\lambda ij} := \{x \in B_i : f_{ij}(x) \in (A_\lambda)_x\}.$$

And for $0 \leq j \leq i$ define

$$D_{\lambda ij} := \{x \in B_i : (f_{ij}(x), f_{ij+1}(x)) \subseteq (A_\lambda)_x\}.$$

It is not immediately clear what these sets are defining.

The $C_{\lambda ij}$ gives the first m components of the lines in the above Figure 2.5 and the $D_{\lambda ij}$ gives the first m components of the spaces between these lines. I will show more formally why these sets are important later.

Now we see that each B_i, $C_{\lambda ij}$ and $D_{\lambda ij}$ is a definable subset of R^m and there are also only finitely many of them. Hence, by the inductive hypothesis we can take a decomposition, D of R^m which partitions B_i, $C_{\lambda ij}$ and $D_{\lambda ij}$ (I$_m$) such that for every $E \in D$, if $E \subseteq B_i$ then $f_{i1}|E, ..., f_{ii}|E$ are continuous functions (II$_m$).

Now for each cell $E \in D$ set

$$D_E := \{(f_{i0}|E), ..., (f_{ii+1}|E), \Gamma(f_{i1}|E), ..., \Gamma(f_{ii}|E)\}.$$

Where i is such that $E \subset B_i$. (Note that we can do this as D partitions the B_i.)

Then $D^* := \cup\{D_E : E \in D\}$ is a decomposition of R^{m+1} by Remark 2.3.9.

Van den Dries notes that D^* partitions $A_1, ..., A_k$. Why is this the case?

Well, lets take some $H \in D^*$. Then $H \in D_E$ for some $E \in D$. We want to show that for any λ either $H \subseteq A_\lambda$ or $H \cap A_\lambda = \emptyset$. Suppose $E \subseteq B_i$. There are two cases to consider:

<u>Case 1</u> $H = (f_{ij}|E, f_{ij+1}|E)$ for some $0 \leq j \leq i+1$. Suppose that $H \cap A_\lambda \neq \emptyset$ and let $h \in H \cap A_\lambda$.

Then $h = (x, r)$ for some $x \in E$ such that $f_{ij}(x) < r < f_{ij+1}(x)$ i.e. $x \in E \cap D_{\lambda ij}$ by construction of the functions f_{ij}.

Take an arbitrary element $y \in H$. Then $y = (x', r')$ for some $x' \in E$ such that $f_{ij}(x') < r' < f_{ij+1}(x')$.

Suppose for a contradiction that $y \notin A_\lambda$. Then $x' \notin D_{\lambda ij}$. This is a contradiction as D partitions the sets $D_{\lambda ij}$.

<u>Case 2</u> $H = \Gamma(f_{ij}|E)$ for some $1 \leq j \leq i$. The proof for this case is similar as above but instead we use the fact that D partitions the $C_{\lambda ij}$.

This concludes the proof of (I_{m+1}).

Now for (II_{m+1}).

We will use the following result.

Lemma 2.3.13. *[3, p. 56] Let X be a topological space, $(R_1, <)$, $(R_2, <)$ dense linear orderings without endpoints and $f : X \times R_1 \to R_2$ a functions such that for each $(x, r) \in X \times R_1$*

(i) $f(x, \cdot) : R_1 \to R_2$ is continuous and monotone on R_1,

(ii) $f(\cdot, r) : X \to R_2$ is continuous at x.

Then f is continuous.

Proof. The proof here does not add to the comprehension of the proof of the cell decomposition theorem so I will omit it. □

Let $f : A \to R$ be a definable function on a definable set $A \subseteq R^{m+1}$. We show that there is a decomposition with f continuous on each cell that is contained in A.

If we can partition A into finitely many definable sets $A_1, ..., A_k$ such that f is continuous on each A_i then by (I_{m+1}) there is a decomposition, D, of R^{m+1} partitioning each A_i. Hence, for every cell $B \in D$ if $B \subseteq A$ then $B \subseteq A_i$ for some i. Hence f is continuous on B and we are done.

Now, again due to (I_{m+1}), it is possible to partition A into finitely many cells. Hence, if we prove the result on an arbitary cell contained in A then we are done. For simplicity we may just assume that A is a cell.

We first consider the case where A is not open. So, A is a $(i_1, ..., i_{m+1})$-cell where $i_j = 0$ for some $1 \leq j \leq m + 1$.

Then consider the homeomorphism (defined in 2.3.3), $p_A : A \to p(A)$. Then as A is not open, $p(A) \subseteq R^n$ for some $n \leq m$. Then by the inductive hypothesis (II_m) $p(A)$ can be partitioned into finitely many definable sets, $B_1, ..., B_l$ such that $(f \circ p_A^{-1})|B_j$ is continuous for each j. Therefore, $p_A^{-1}(B_1), ..., p_A^{-1}(B_l)$ partitions A and $f|p_A^{-1}(B_j)$ is continuous as required.

Now consider the case that A is an open cell. We first define the notion of a function being *well-behaved*. We say that a function$f$ is **well-behaved** at a point $(p, r) \in A$ if $p \in C$ for some box (i.e. cartesian product of intervals) $C \subseteq R^m$ and $a < r < b$ for some $a, b \in R$ such that

(i) $C \times (a, b)$ is contained in A

(ii) for all $x \in C$ $f(x, \cdot)$ is continuous and monotone on (a, b)

(iii) $f(\cdot, r)$ is continuous at p

We define A^* to be the set of all points in A where f is well-behaved. We also note that A^* is definable.

Van den Dries gives and proves the claim that A^* is dense in A. Again as this does not add much to the understanding of the theorem I will leave out this proof.

As we are assuming I_{m+1} we can take a decomposition, D, of R^{m+1} partitioning A and A^*. Consider a cell, $E \in D$ contained in A. We must show that f is continuous on E, or that we may take a decomposition of R^{m+1} partitioning E where f is continuous on each cell of this new decomposition. Van den Dries does not mention this, but if E is not open then we can use the same argument as before using the function p (from 2.3.3) to show that we can partition E into finitely many cells on which f is continuous. So we may suppose that E is open.

As $E \subseteq A$ and A^* is dense in A then $E \cap A^* \neq \emptyset$ (See aside on topology). Further, as D partitions A^* then $E \subseteq A^*$. Then f is well-behaved at every point in E, so, for every point $(p, r) \in E$, $f(\cdot, r)$ is continuous at p and every point in E is contained in some box $C \times (a, b)$. Moreover, every such box is contained in E. Hence, E is a union of boxes $C \times (a, b)$ satisfying conditions (i),(ii),(iii). Now, if we take $X = C$, $R_1 = (a, b)$ and $R_2 = R$ in Lemma 2.3.13 then we see that f is continuous on each box $C \times (a, b)$ and therefore f is continuous on D.

This completes the proof of II_{m+1} and of the cell decomposition theorem.

□

To see that the assumption of o-minimality is appropriate for the cell decomposition theorem consider the structure $\mathcal{R} = \langle \mathbb{R}, <, \sin \rangle$. This is not o-minimal as the set $A = \{y \in \mathbb{R} : \sin(y) = 0\} \subseteq \mathbb{R}$ is definable in $\mathcal{R}$ and is not a union of finitely many intervals and points. A is a union of infinitely many points in $\mathbb{R}$, hence a decomposition of $\mathbb{R}$ into cells which partitions A does not exist, as one would require an infinite number of (0)-cells.

I will now prove a strengthened version of the cell decomposition, given as an exercise here [3, p. 58].

First a quick proposition.

Proposition 2.3.14. *Let A be an i-cell where $i = (i_1, ..., i_m)$. Let $k = i_1 + ... + i_m$ and let $\lambda(1) < ... < \lambda(k)$ be the indices for which $i_{\lambda(j)} = 1$, where $1 \leq j \leq k$. If A is definable in $\mathcal{R} = \langle R, < \rangle$ then $p_i | A : A \to p(A)$ (see 2.3.3 for definition) is definable in $\mathcal{R}$.*

Proof.

$$(x_1, ..., x_m, y) \in \Gamma(p_i|A) \iff (x_1, ..., x_m) \in A \;\&\; p_i(x_1, ..., x_m) = y$$
$$\iff x \in A \;\&\; (x_{\lambda(1)}, ..., x_{\lambda(k)}) = y.$$

□

Proposition 2.3.15. *Below, unless specified, 'definable' means definable in some given o-minimal structure.*

(I_m^{def}) If the sets $A_1, .., A_k \subseteq R^m$ are definable, then there is a decomposition of R^m partitioning each set A_i, all of whose cells are definable in the model-theoretic structure $\langle R, <, A_1, ..., A_k\rangle$.

(II_m^{def}) Let the function $f : A \to R$, $A \subseteq R^m$, be definable. Then there is a decomposition D of R^m partitioning A, such that the restriction $f|B$ to each cell $B \in D$ with $B \subseteq A$ is continuous, and each cell in D is definable in the model-theoretic structure $\langle R, <, \Gamma(f)\rangle$.

Proof. (I_m^{def}) We prove by induction and begin by proving I_1^{def}. Let $A_1, .., A_k \subseteq R$ be definable. Then, from o-minimality, each A_i is a finite union of cells. First, we show that a specific element a is definable in $\mathcal{R} = \langle R, <, A_1, ..., A_k\rangle$. We define a to create a left-hand interval $(-\infty, a)$ which, given any i, is either contained in or disjoint from A_i. $x = a \iff$ for each i the following holds, for all y with $-\infty < y < a$, $y \notin A_i$ or for all y if $-\infty < y < a$ then $y \in A_i$ and a is the largest such element which has this property.

Then define the cells C_0 by $x \in C_0 \iff x \in (-\infty, a)$ and C_1 by $x \in C_1 \iff x = a$.

We then repeat this process for $\bigcup_{i=0}^k A_i - (-\infty, a]$ where in some sense $-\infty$ is replaced by a. This gives the required cell decomposition.

Induction Hypothesis Let I_n^{def} hold for all $n \leq m$. Consider definable sets $A_1, ..., A_k \subseteq R^{m+1}$. We will refer to the cell decomposition proof. Let B_i, $C_{\lambda ij}$ and $D_{\lambda ij}$ be as in the proof of the cell decomposition, with $i \in \{0, ..., M\}$, $\lambda \in \{1, ..., k\}$ and $j \in \{1, ..., i\}$. Then by the induction hypothesis there is a decomposition, D, of R^m into cells partitioning each B_i, $C_{\lambda ij}$ and $D_{\lambda ij}$ such that the cells in the decomposition are definable in $\langle R, <, (B_i), (C_{\lambda ij}), (D_{\lambda ij})\rangle$ (*). (Note the notation here, I've written (B_i) to denote $B_0, ..., B_M$ etc..). Then D^* (as in the original proof) is a decomposition of R^{m+1} which paritions $A_1, ..., A_k$. Let $H \in D^*$ then there are two cases to consider 1) $H = (f_{ij}|E, f_{ij+1}|E)$ or 2) $H = \Gamma(f_{ij}|E)$ for some function f_{ij} and $E \in D$. In case 1),

$$(x, y) \in H \iff x \in E \ \&\ f_{ij}(x) < y < f_{ij+1}(x).$$

In case 2),

$$(x, y) \in H \iff x \in E \ \&\ f_{ij}(x) = y.$$

Note that each fiber $(A_\lambda)_x$ is definable in $\mathcal{R} = \langle R, <, A_1, ..., A_k\rangle$ and so $\text{bd}((A_\lambda)_x)$ is definable in $\mathcal{R}$. Hence every $\text{bd}_m(A_\lambda)$ is clearly definable in $\mathcal{R}$ and so Y is definable in $\mathcal{R}$. (For the definitions of Y and $\text{bd}_m(A_\lambda)$ see the proof of the cell decomposition). This gives that each fiber Y_x is definable in $\mathcal{R} \Rightarrow$ every B_i is definable in $\mathcal{R} \Rightarrow$ every f_{ij} and $C_{\lambda ij}$, $D_{\lambda ij}$ is definable in $\mathcal{R}$. Hence, as E is definable in $\langle R, <, (B_i), (C_{\lambda ij}), (D_{\lambda ij})\rangle$ by (*), then E is definable in $\mathcal{R}$.

Hence, E, f_{ij} and f_{ij+1} are definable in $\mathcal{R}$ so H is definable in $\mathcal{R}$ in either case.

(II_m^{def}) Again, we prove by induction and begin by proving II_1^{def}. Let $\mathcal{R} = \langle R, <, \Gamma(f)\rangle$. A is a finite union of intervals and points so we apply the monotonicity theorem to each interval contained in A. This gives $a_1 < ... < a_k$ such that if $(a_i, a_{i+1}) \subseteq A$ then f is continuous on (a_i, a_{i+1}). Also, if $\{a_i\} \subseteq A$ then f is continuous on $\{a_i\}$. Now, using Proposition 2.2.8 we can take $a_1, ..., a_k$ such that they are definable in $\mathcal{R}$.

Let $D = \{(-\infty, a_1), ..., (a_k, +\infty), \{a_1\}, ..., \{a_k\}\}$. Then D is a cell decomposition of R, every cell in D is definable in $\mathcal{R}$ and f is cellwise continuous with respect to this decomposition as required.

Induction Hypothesis Let II_n^{def} hold for all $n \leq m$. Again, we will refer to the cell decomposition proof. In particular, we suppose that A is a cell. If A is not open then consider $p(A) \subseteq R^n$ where $n < m + 1$, this is the map from 2.3.3. Then by II_n^{def} and that fact that $p(A)$ and $f \circ p_A^{-1}$ are definable, there is a decomposition, D, of R^m into cells that partitions $p(A)$ such that for all $B \in D$ if $B \subseteq p(A)$ then $f \circ p_A^{-1}|B$ is continous and each $B \in D$ is definable in $\langle R, <, \Gamma(f \circ p_A^{-1})\rangle$ (*). Let $D = \{B_1, ..., B_l\}$.

Now the restriction of f to $p_A^{-1}(B)$ is continuous for each $B \in D$ where $p_A^{-1}(B) \subseteq A$. By I_{m+1}^{def} we can partition each $p_A^{-1}(B_i)$ by an appropriate cell decomposition, D', where each cell in the decomposition is definable in $\langle R, <, p_A^{-1}(B_1), ..., p_A^{-1}(B_l)\rangle$ (+). Let $D' = \{C_1, ..., C_t\}$. Hence, if $C_i \subseteq A$ then $C_i \subseteq p_A^{-1}(B)$ for some $B \in D$, hence f is continuous on C_i.

We now show that $f \circ p_A^{-1}$ is definable in $\mathcal{R} = \langle R, <, \Gamma(f)\rangle$. Note again that A is definable in $\langle R, <, \Gamma(f)\rangle$. $(x, y) \in \Gamma(f \circ p_A^{-1}) \iff$ there is some $a \in A$ such that $p_A(a) = x$ and $f \circ p_A^{-1}(x) = y \iff$ there is some $a_1, ..., a_{m+1} \in R$ such that $(a_1, ..., a_{m+1}) \in A$ and $(a_{\lambda(1)}, ..., a_{\lambda(n)}) = x$ and $(a_1, ..., a_{m+1}, y) \in \Gamma(f)$. Note that the choice of $\lambda(i)$ is given by A, as in 2.3.14.

Hence, $f \circ p_A^{-1}$ is definable in $\mathcal{R}$. By (*), this implies that every $B_i \in D$ is definable in $\mathcal{R}$. Now for each $B_i \subseteq p(A)$, $(x_1, ..., x_{m+1}) \in p_A^{-1}(B_i) \iff$ there exists $y \in B_i$ such that $(x_{\lambda(1)}, ..., x_{\lambda(n)}) = y$. Hence, for each $B_i \subseteq p(A)$, $p_A^{-1}(B_i)$ is definable in $\mathcal{R}$. Hence, by (+) every C_i is definable in $\mathcal{R}$ as required.

Now suppose that A is open. Let A^* be the set containing all points in A at which f is *well-behaved* (see the proof of cell decomposition for this definition). We show that A^* is definable in $\mathcal{R}$. $(p, r) \in A^* \iff (p, r) \in A$ and there exists some box $C \subseteq R^m$ and $a, b \in R$ with $a < r < b$ such that $C \times (a, b) \subseteq A$ and for all $y \in C$, $f(y, \cdot)$ is continuous and monotone on (a, b) and $f(\cdot, r)$ is continuous at p. This shows informally that A^* is definable in $\mathcal{R}$.

Now, by I_{m+1}^{def}, take a decomposition, D'', of R^{m+1} that partitions A and A^* such that every cell in D'' is definable in $\langle R, <, A, A^*\rangle$. As is the orginial proof, this decomposition satifies that f be cellwise continuous. Also A and A^* are definable in $\mathcal{R}$, hence every cell in D'' is definable in $\mathcal{R}$, giving us the required decomposition.

□

3. Curve Selection

3.1. Introduction

In this chapter I will look at a particular property, *curve selection*, which holds for a particular class of o-minimal structures - expansions of ordered abelian groups. In order to show that curve selection holds we first begin by showing that it is possible to pick an element from a definable set, in a definable way, this then quickly leads to *definable choice*. A nice way to understand definable choice is that it tells us that for certain relations, there exists definable functions which 'pick' instances such that the relations hold. From these results we see that curve selection follows. This is a very useful result as normally we have no notion of distance and so cannot use the usual ideas of points in the closure of a set being the limits of sequences. Curve selection, however, provides us with a means of deducing that an element is in the closure of a set, saying that an element, x, is contained in the closure of a set if it is a limit point of a definable continuous injective map from an interval to the set.

I will then give an application of this fact in order to show that the image of a continuous definable map of a closed and bounded set is itself closed and bounded. From this we can also deduce a fixed point theorem for definable functions in o-minimal expansions of ordered abelian groups and I will show this.

I will also introduce a strict notion of dimension which will be useful in order to show that curve selection does not hold for arbitrary o-minimal structures.

3.1.1. Preliminaries

Before beginning this chapter it will be necessary to describe what is meant by an *expansion*. Van den Dries defines this notion by way of example, giving the example of an ordered abelian group [3, p. 23]. I will give this example and a further example that will be useful in later chapters.

Consider an o-minimal structure $(R, <, \mathcal{S})$. We say that $(R, <, \mathcal{S})$ expands an ordered abelian group if there are is a zero element 0 in R and functions $- : R \to R, + : R^2 \to R$ belonging to $\mathcal{S}$ such that $(R, <, 0, -, +)$ is an ordered abelian group.

Similarly $(R, <, \mathcal{S})$ expands the ordered field of real numbers if $R = \mathbb{R}$ (clearly containing 0 and 1) and the functions $+, \cdot : R^2 \to R$ belong to $\mathcal{S}$ such that $(R, <, 0, 1, +, \cdot)$ is a field.

So, to say that an o-minimal structure expands some algebraic structure just means that the o-minimal structure contains all the operations of the algebraic structure.

For the rest of this chapter we fix an o-minimal structure $(R, <, \mathcal{S})$ which expands an ordered abelian group $(R, <, 0, -, +)$.

We also set

$$|x| := \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0 \end{cases}$$

for $x \in R$ and $|x| := \max\{|x_1|, ..., |x_m|\}$ for $x = (x_1, ..., x_m) \subseteq R^m$ where $m > 0$.

3.2. Curve Selection

3.2.1. Dimension

The notion of dimension will be useful for proofs in this section. I will give the definition and a useful result given by van den Dries in chapter 4 of his book.

Definition 3.2.1. *[3, p. 63] We define the* ***dimension*** *of a nonempty definable set $X \subseteq R^m$ by*

$$dim X := max\{i_1 + ... + i_m : X \text{ contains an } (i_1, ..., i_m)\text{-cell}\}.$$

To the empty set we assign dimension $-\infty$.

In particular, if X is an $(i_1, ..., i_m)$-cell then $\dim(X) = i_1 + ... + i_m$.

Below I will give a useful result and provide a sketch of the proof. This result tells us that definable bijections preserve dimension. Thus if two definable subsets have different dimension then we can conclude that there can't be a definable bijection between them.

Proposition 3.2.2. *[3, p. 64] If $X \subseteq R^m$ and $Y \subseteq R^n$ are definable and there is a definable bijection between X and Y, then dim$X$ =dim$Y$.*

Proof. This proof provides a nice example of using cell decomposition. I will just give an outline of the proof as given by van den Dries [3, p. 64] and give more detail regarding the step using cell decomposition as a detailed explanation is not given by van den Dries.

Let dim$X = d$ and dim$Y = e$. We consider a bijection $f : X \to Y$ and show that $d \leq e$. $e \leq d$ then follows from the inverse of f an hence $d = e$. We take a cell A in X with dim$A = d$. Use the map from definition 2.3.3 to transform A into an open cell, $p(A)$. Then dim$p(A) =$ dim$(A) = d$. Hence, if dim$p(A) \leq$ dim$f(A)$ then $d \leq e$. In particular, we may assume that A is an open cell in R^d and that $Y = f(A)$. We take a decomposition of R^n which partitions Y (using the cell decomposition theorem). Let $Y = C_1 \cup ... \cup C_k$. Then $f^{-1}(Y) = f^{-1}(C_1) \cup ... \cup f^{-1}(C_k) = A$. Now, here van den Dries just notes that by the cell decomposition theorem for some $1 \leq i \leq k$, $f^{-1}(C_i)$ contains an open cell, B. Why is this the case?

Well, suppose that there is no open cell in $f^{-1}(C_i)$ for any i. As each $f^{-1}(C_i)$ is definable there is a decomposition of R^d into cells partitioning each $f^{-1}(C_i)$, by the cell decomposition theorem. Also, each cell in this decomposition contained in some $f^{-1}(C_i)$ is not open, i.e. each of these cells has empty interior. Now, by note 2.5 of van den Dries book [3, p. 50], we know that if A is a union of finitely many non-open cells in R^m then A has empty interior. This is a contradiction as A is open.

Now let C_i be a $(j_1, ..., j_n)$-cell. We then consider a map $B \to C_i \to p(C_i)$ in order to show that $d \leq j_1 + ... + j_n$ and hence $d \leq e$. To see details see [3, pp. 63-64]. □

3.2.2. Definable Choice & Curve Selection

First we consider some properties of such structures.

Due to the group structure we can definably pick an element $e(X) \in X$ from each non-empty definable set X. By *definably*, we mean that given a non-empty definable set X we can give a formula defining $\{e(X)\}$.

This seems trivial, so to get some idea of why it may not be consider a naive way of constructing $e(X)$. First, let's say we define $e(X)$ to be the least element of X, well this may not exist i.e. if X is an interval.

Suppose then we take $e(X)$ as some point greater than the infimum of X, then it may be the case that $e(X) \notin X$ for $X = \{r\} \cup (a, b)$ with $r < a$.

The idea then for $X \subseteq R$ is to take $e(X)$ to be the least element if X has one or the midpoint of the left-most bounded interval if X has one. This is given more formally below and we also define $e(X)$ inductively for $X \subseteq R^m$ for $m \geq 1$.

Definition 3.2.3. *[3, p. 93]*

(i) Let $X \subseteq R$ be definable and nonempty. If X has a least element, then let $e(X)$ be this least element. If X does not have a least element, let (a, b) be its left-most interval: $a =$ inf$X$, $b =$ sup$\{x \in R : (a, x) \subseteq X\}$. Then $a < b$ and $(a, b) \subseteq X$; now set

$$e(X) := \begin{cases} 0 & \text{if } a = -\infty, b = +\infty, \\ b - 1 & \text{if } a = -\infty, b \in R, \\ a + 1 & \text{if } a \in R, b = +\infty, \\ (a + b)/2 & \text{if } a, b \in R. \end{cases}$$

(ii) Let $X \subseteq R^m$ be definable and nonempty, $m > 1$, and let $\pi : R^m \to R^{m-1}$ be the projection on the first $m - 1$ coordinates. Then $\pi(X) \subseteq R^{m-1}$ so we may assume inductively that an element $a = e(\pi(X))$ of $\pi(X)$ has been defined. Then $X_a \subseteq R$ and we put $e(X) := (a, e(X_a))$.

Note that there is typo here in [3] when defining b and we have replaced the R with an X in 'sup$\{x \in R : (a, x) \subseteq R\}$'.

Clearly $e(X) \in X$ in case (i). To see that $e(X) \in X$ in case (ii), note that $a \in \pi(X)$ and that $e(X_a) \in X_a = \{x \in R : (a, x) \in X\}$.

Proposition 3.2.4 (Definable Choice)**.** *[3, p. 94]*

(i) If $S \subseteq R^{m+n}$ is definable and $\pi : R^{m+n} \to R^m$ the projection on the first m coordinates, then there is a definable map $f : \pi(S) \to R^n$ such that $\Gamma(f) \subseteq S$.

(ii) *Each definable equivalence relation on a definable set X has a definable set of representatives.*

Note: We think of an equivalence relation on X as a set of ordered pairs of elements of X, so we can say the equivalence relation is definable if this set is definable in the usual sense.

What definable choice says is that if for every $t \in T \subseteq R^m$ there is at least one $x \in X \subseteq R^n$ such that $R(t,x)$ holds for some definable relation R then there is a definable map $f : T \to X$ that assigns (i.e chooses) exactly one $x \in X$ to each $t \in T$ such that the relation holds. (It may be satisfying - or not! - to note the similarity with the axiom of choice - that for any set of non-empty sets there is a choice function which chooses an element from each set).

Van den Dries notes that (i) is the property of having definable Skolem functions - which allow the removal of existential quantifiers in place of these functions. Another example of this property comes from Peano Arithmetic, as every nonempty definable set in a model of Peano Arithmetic has a least member, hence - using the above example - we can pick the least x such that $R(t,x)$ holds.

Proof. (i) Van den Dries just notes that such a definable map is $f(x) = e(S_x)$ for $x \in \pi(S)$. Why does this work? Note that $\Gamma(f) = \{(x, e(S_x) : x \in \pi(S)\}$ and $e(S_x) \in S_x$ for each x.

(ii) Van den Dries gives the definable set $\{e(A) : A \text{ is an equivalence class}\}$ which is clearly a set of representatives.

□

We now give the following lemma which tells us that points in the closure of a definable set X but not in X are limit points of definable paths in X (It is clear that any point in X is already a limit point of a definable path in X).

Lemma 3.2.5 (Curve Selection)**.** *[3, p. 94] If $a \in cl(X) - X$, where X is definable, then there is a definable continuous injective map $\gamma : (0,\epsilon) \to X$, for some $\epsilon > 0$, such that $lim_{t\to 0}\gamma(t) = a$.*

Proof. This is the proof as given by van den Dries, though I will give some more detail.

Suppose $a \in \text{cl}(X)$, then we can take elements in X as close as we want to a. Hence, the set $\{|a - x| : x \in X\}$ contains an interval $(0, \epsilon')$ for arbitrarily small $\epsilon' > 0$. Also, note that this set is definable. For every $t \in (0, \epsilon')$ there is an $x \in X$ such that $|a - x| = t$ (*).

We now use definable choice. van den Dries just states that by definable choice there is a definable function $\gamma : (0, \epsilon') \to X$ such that $|a - \gamma(t)| = t$ for all $t \in (0, \epsilon')$. To see more explicitly why this is the case, consider the definable set $S = \{(x, y) \in R \times X : x \in (0, \epsilon'),\ x = |a - y|\}$. Let π be the projection on the first m coordinates, i.e. $\pi(S) = (0, \epsilon')$, this comes from (*). Then by definable choice there is a definable map $\gamma : (0, \epsilon') \to R$ such that $\Gamma(\gamma) \subseteq S$. i.e.

$$\Gamma(\gamma) = \{(x, \gamma(x)) : x \in (0, \epsilon')\} \subseteq \{(x, y) \in R \times X : x \in (0, \epsilon'), x = |a - y|\}$$

$\Rightarrow$ for all $x \in (0, \epsilon')$, $x = |a - \gamma(x)|$ and $\gamma(x) \in X$ as required.

To finish the proof we note that by the monotonicity theorem we can take $0 = \epsilon_1 < ... < \epsilon_k = \epsilon'$ such that γ is continuous on each interval $(\epsilon_i, \epsilon_{i+1})$. In particular, γ is continuous on $(0, \epsilon_2)$. Let $\epsilon_2 = \epsilon$. Cleary γ is injective on $(0, \epsilon)$. To see this take $x, x' \in (0, \epsilon)$ with $\gamma(x) = \gamma(x')$. Then $x = |a - \gamma(x)| = |a - \gamma(x')| = x'$. Also, it is clear that $\lim_{t \to 0} \gamma(t) = a$. □

Van den Dries gives the following example as an exercise and states that it shows that definable curve selection fails for the o-minimal structure $(\mathbb{R}, <)$. I will give a solution to this exercise and explain why this shows that curve selection fails as this point is not immediately clear.

Example 3.2.6. [3, p. 98] Consider the o-minimal model-theoretic structure $\langle \mathbb{R}, < \rangle$ and the set

$$X := \{(x, y) \in \mathbb{R}^2 : 0 < x < 1, 0 < y < 2\},$$

which is definable in $\langle \mathbb{R}, < \rangle$ using the constants 0, 1 and 2. Note that $(1, 2) \in \text{cl}(X) - X$ and show that there is no subsets Y of X such that Y is definable in $\langle \mathbb{R}, < \rangle$ using constants $\dim(Y) = 1$ and $(1, 2) \in \text{cl}(Y)$.

<u>Solution</u> Let $Y \subseteq X \subseteq \mathbb{R}^2$ be definable in $\langle \mathbb{R}, < \rangle$ using constants and $\dim(Y)$=1. $\langle \mathbb{R}, < \rangle$ is o-minimal, so we can take a cell decomposition of $\mathbb{R}^2$ which partitions Y. Hence, Y is a union of finitely many cells, $Y = C_1 \cup ... \cup C_k$ say. As $\dim(Y) = 1$, Y

must not contain a $(1,1)$-cell and must contain a $(0,1)$-cell or a $(1,0)$-cell. Suppose $(1,2) \in \text{cl}(Y) - Y$ then $(1,2)$ is a limit point of Y. Hence for all $a_1, b_1, a_2, b_2 \in \mathbb{R}$ if $a_1 < 1 < b_1$ and $a_2 < 2 < b_2$ then $(a_1, b_1) \times (a_2, b_2) \cap Y \neq \emptyset$. Suppose for no C_i, $(1,2) \in \text{cl}(C_i) - C_i$. Then for every C_i there is some $a_1, b_1, a_2, b_2 \in \mathbb{R}$ with $a_1 < 1 < b_1$ and $a_2 < 2 < b_2$ such that for all $x_1, y_1, x_2, y_2 \in \mathbb{R}$ if $a_1 < x_1 < 1 < y_1 < b_1$ and $a_2 < x_2 < 2 < y_2 < b_2$ then $(x_1, y_1) \times (x_2, y_2) \cap C_i = \emptyset$. Consider such a_1, b_1, a_2, b_2 for each C_i and take the largest such a_1 and a_2 and the least such b_1 and b_2 (we can do this as $k \in \mathbb{N}$), call them a_1', a_2', b_1', b_2'. Then for every C_i and for all $x_1, y_1, x_2, y_2 \in \mathbb{R}$ if $a_1' < x_1 < 1 < y_1 < b_1'$ and $a_2' < x_2 < 2 < y_2 < b_2'$ then $(x_1, y_1) \times (x_2, y_2) \cap C_i = \emptyset$. Hence, $(x_1, y_1) \times (x_2, y_2) \cap Y = \emptyset$ which is a contradiction. Hence, $(1,2) \in \text{cl}(C) - C$ for some cell C in Y.

If C is a $(0,0)$-cell then clearly this does not hold as $(1,2) \in \text{cl}(C) \iff C = \{(1,2)\}$. If C is a $(0,1)$-cell then C is a vertical line with $(1,2)$ as a limit point, however no such line is contained in $Y \subseteq X$. If C is a $(1,0)$-cell then $C = \{(x, f(x)) : x \in (a,b) \subseteq (0,1)\}$ for some definable function f and interval (a,b), however the only definable functions in $\langle \mathbb{R}, < \rangle$ are constant functions or coordinate functions $(x_1, ..., x_m) \mapsto x_i$ or combinations of the two (see [3, p. 24]), for example,

$$f(x) = \begin{cases} x & \text{if } x \in (a, a'] \\ 2 & \text{if } x \in (a', b) \end{cases}$$

Hence if $(1,2) \in \text{cl}(C)$ then $b = 1$ and f must be the constant function $f(x) = 2$ on some interval $(a', b) \subseteq (a, b)$. Hence, C is not contained in X. Hence, no such Y exists. $\square$

Now, how does this example show that definable curve selection fails for $(\mathbb{R}, <)$? Well, take X as above and note that $(1,2) \in \text{cl}(X) - X$ then suppose for a contradiction that there is a definable continuous injective function $\gamma : (0, \epsilon) \to X$ such that $\lim_{t \to 0} \gamma(t) = (1,2)$. Then $\Gamma(\gamma) = \{(x, \gamma(x)) : x \in (0, \epsilon)\}$ is definable. Take π to be the projection map on the last 2 coordinates. Then $Y := \pi(\Gamma(\gamma)) = \{\gamma(x) : x \in (0, \epsilon)\} \subseteq X$ is definable. Further, there is a definable bijection from $(0, \epsilon)$ to Y, namely γ. $\dim(0, \epsilon) = 1$. Hence, by Proposition 3.2.2, $\dim(Y) = 1$.

Cleary, $(1,2) \in \text{cl}(Y)$. So we have constructed a subset Y of X that is definable in $(\mathbb{R}, <)$, where $\dim(Y) = 1$ and $(1,2) \in \text{cl}(Y)$. This contradicts the above example.

Hence, as noted by van den Dries, the assumption that we are working with an expansion of an ordered group is appropriate.

3.2.3. An Application

I will now work towards a nice result given by van den Dries that the image of a closed bounded definable set under a continuous definable function is closed and bounded.

First it will be necessary to define the notion of *bounded* used here.

Definition 3.2.7. *[3, p. 95] We call a set $A \subseteq R^m$ **bounded** if for some $r \in R$ $|a| < r$ for all $a \in A$.*

This is saying that a set A is bounded if A is bounded (in the usual sense) in each coordinate.

The following lemma tells us that the closure of a bounded cell under the projection map is equal to the closure of the projection of the cell. This result is then used to prove the main result of this section, also I will include the proof as it makes use of the monotonicity theorem and curve selection.

Lemma 3.2.8. *[3, p. 95] Let C be a bounded cell in R^m, $m > 1$, and $\pi : R^m \to R^{m-1}$ the projection on the first $m-1$ co-ordinates. Then $\pi(cl(C)) = cl(\pi(C))$.*

Proof. There are two cases to consider, either $C = (f, g)_{\pi(C)}$ or $C = \Gamma(f)$ for some appropriate functions f and g. Van den Dries leaves the case $C = \Gamma(f)$ to the reader so I will do this case. Let $f \in C(\pi(C))$.

Firstly, note that π is a continuous function. (To see this: Let $V \subseteq R^{m-1}$ be an open set then the preimage $\pi^{-1}(V) = \{x \in R^m : \pi(x) \in V\} = V \times R$ is open in R^m). Then as a basic fact from topology, $\pi(\text{cl}(C)) \subseteq \text{cl}(\pi(C))$.

Let $a \in \text{cl}(\pi(C))$. Need to find $s \in R$ such that $(a, s) \in \text{cl}(C)$. If $a \in \pi(C)$ then there is s such that $(a, s) \in C \subseteq \text{cl}(C)$ so we are done.

Suppose that $a \notin \pi(C)$. Then by curve selection there is a continuous definable function $\gamma : (0, \epsilon) \to \pi(C)$ such that $\lim_{t \to 0} \gamma(t) = a$.

As C is bounded there is an $r \in R$ with $r > 0$ such that $-r < f(x) < r$ for all $x \in \pi(C)$. Define $\lambda : (0, \epsilon) \to R$ by $\lambda(t) = f(\gamma(t))$. Then as f is bounded by r, $-r < \lambda(t) < r$ for all $t \in (0, \epsilon)$. By applying the monotonicity theorem to λ we know there is $0 < \sigma < \epsilon$ such that λ is either constant or strictly monotone (and continuous) on $(0, \sigma)$.

Clearly if λ is constant on $(0, \sigma)$, say $\lambda(t) = s$, then taking $U = (-\sigma, \sigma)$ we see that $\lambda(U \cap (0, \epsilon) - \{0\}) = \{s\} \subseteq V$ where V is any open neighbourhood of s. Hence $\lim_{t \to 0} \lambda(t)$ exists. Also, if λ is strictly monotone on $(0, \sigma)$ then consider the image $\lambda((0, \sigma))$. This is a definable subset of R. Hence by Lemma 1.4.5 (i), $\lambda((0, \sigma))$ has an infinum and supremum in R_∞. Further $\lambda((0, \sigma))$ is bounded by r, hence its infimum and supremum exist in R. Consider the case where λ is strictly increasing on $(0, \sigma)$. Then $\inf \lambda((0, \sigma)) = s$ is the required limit point as $t \to 0$. To see this, take an open neighbourhood V of s. Then $U = \lambda^{-1}(V) \cup (-\sigma, \sigma)$ is an open neighbourhood of 0 (as λ is continuous) and gives $\lambda(U \cap (0, \epsilon) - \{0\}) = \lambda(\lambda^{-1}(V) \cap (0, \epsilon) - \{0\}) \subseteq V$.

Hence the function $g : (0, \epsilon) \to C, t \mapsto (\gamma(t), \lambda(t))$ is a continuous definable function and $\lim_{t \to 0} g(t) = (a, s)$. Hence $(a, s) \in \text{cl}(C) \Rightarrow a \in \pi(\text{cl}(C)) \Rightarrow \text{cl}(\pi(C)) \subseteq \pi(\text{cl}(C))$. □

Remark 3.2.9. This result easily extends to the projection map $R^m \to R^{m-n}$ on the first $m - n$ coordinates. To see this note that $\pi(\text{cl}(C)) = \text{cl}(\pi(C)) \Rightarrow \pi \circ \pi(\text{cl}(C)) = \pi \text{cl}(\pi(C)) = \text{cl}(\pi \circ \pi(C))$.

The proof that the image of a closed bounded set is bounded is short and again uses the monotonicity theorem as in the above proof to establish the existence of limits. Hence I will just state this as a lemma and not give a proof.

Lemma 3.2.10. *[3, p. 95] Let $f : X \to R^n$ be a definable continuous map on a closed bounded set $X \subseteq R^m$. Then $f(X)$ is bounded in R^n.*

Proof. Omitted. □

Lemma 3.2.11. *[3, p. 96] If $f : X \to R^n$ is a continuous definable function on a closed bounded set $X \subseteq R^m$, then $f(X)$ is closed and bounded in R^n.*

Proof. The proof given by van den Dries is already very clear and concise but also shows why 3.2.8 is useful, so I will give a brief description of it.

Take Y to be the reversed graph of f i.e. $Y := \{(f(x), x) : x \in X\} \subseteq R^{n+m}$. Then take a cell decomposition of R^{m+n} which partitions Y. Let $Y = C_1 \cup ... \cup C_k$ for cells $C_1, ..., C_k$.

Note that Y is closed, hence $Y = \text{cl}(Y) = \text{cl}(C_1 \cup ... \cup C_k) = \text{cl}(C_1) \cup ... \cup \text{cl}(C_k)$ as the closure of a finite union is equal to the union of the closures of each set. Y is bounded as X and $f(X)$ are bounded, hence each cell is bounded. We can now apply 3.2.8, to see that $\pi(Y) = \pi(\text{cl}(C_1)) \cup ... \cup \pi(\text{cl}(C_1)) = \text{cl}(\pi(C_1)) \cup ... \cup \text{cl}(\pi(C_1)) = f(X)$. Hence $f(X)$ is closed in R^n. □

This result can in fact be generalized to arbitrary o-minimal structures, this was done by Steinhorn and Peterzil - see corollary 2.4 of [8]. In this paper they also establish the result that a definable set, X, is closed and bounded if and only if it is *definably compact* i.e. if every definable continuous map from an interval onto X has both right-hand and left-hand limit points in X [8, p. 770].

The following proposition is an exercise given by van den Dries and gives us a fixed point theorem for functions on closed bounded definable sets.

Proposition 3.2.12 (Fixed Point Theorem)**.** *[3, p. 97] Let $X \subseteq R^m$ be a nonempty closed bounded definable subset of R^m and $f : X \to X$ a definable map such that $|f(x) - f(y)| < |x - y|$ for all distinct points $x, y \in X$. Then f has a unique fixed point.*

Proof. First I will note that van den Dries gives a hint to the solution: Consider points in X where the function $x \mapsto |f(x) - x| : X \to R$ takes its minimum value [3, p. 170].

Consider the map $g : X \to R$ defined by $g(x) = |f(x) - x|$. Then clearly g is a definable function. Further, as f is continuous then so is g. Hence by 3.2.11 $g(X)$ is closed and bounded. As $g(X)$ is a subset of R this means that $g(X) = [a, b]$ for some $a, b \in R$. Let $x \in X$ be such that $a = g(x) = |f(x) - x|$. Suppose for a contradiction that $f(x) \neq x$. Then $|f(f(x)) - f(x)| < |f(x) - x| = a$ which contradicts the minimality of a (as indeed $f(x) \in X$ so $|f(f(x)) - f(x)| \in g(X)$). Hence $x = f(x)$ is a fixed point.

To see that this point is unique suppose that $x, y \in X$ are distinct and $f(x) = x$, $f(y) = y$. Then $|x - y| = |f(x) - f(y)| < |x - y|$. Contradiction. □

4. Exponentiation

4.1. Introduction

A significant development in this field was a result was given by Wilkie [5], saying that the theory of the structure of the real ordered field with exponentiation is model complete - this is similar to quantifier elimination. This result combined with a result due to Khovanskii [9] showed that o-minimal expansions of the real ordered field with exponentiation are o-minimal.

The methods used in proving this are rather complex so I will not consider these. However, I will look at two nice results due to Miller regarding the definability of the exponential function in o-minimal expansions of ordered fields, the growth dichotomys. The first result tells us that for any o-minimal expansion, $\mathcal{R}$, of the ordered field of real numbers, if $\mathcal{R}$ is not *polynomially bounded* then the exponential function is definable in $\mathcal{R}$. The second result gives a more general result applied to arbitrary o-minimal expansions of ordered fields; in this case however we must use the notion of *power boundedness* which differs slightly from the notion of being polynomially bounded.

Note that Miller also proved a growth dichotomy for o-minimal expansions of ordered groups, saying that for $\mathcal{R}$ an o-minimal expansion of an ordered group, $(R, <, +)$, either $\mathcal{R}$ is *linearly bounded* or there is a definable operation $\cdot$ such that $(R, <, +, \cdot)$ is an ordered real closed field [10].

I will only briefly discuss the case for the field of real numbers and give more attention to the more general case. In the general case I will outline the necessary background for the proof of the main theorem of the section and will give an outline of the proof of this. I hope that this can serve as an aid to understanding Miller's paper, giving a concise overview and in places giving more explanation. To see the proof of the theorem in full detail see [4].

In the general case, it will be necessary to properly define what we mean when we speak of the exponential function. We will see that the formal definition satisfies the notion of an exponential function.

4.2. Expansions of the ordered field of real numbers

4.2.1. Preliminaries

Note: By 'ultimately' we will mean 'for all sufficiently large x'.

In this section we fix an o-minimal expansion, $\mathcal{R} = \langle \mathbb{R}, <, 0, 1, +, \cdot, ... \rangle$, of the ordered field of real numbers.

We also need to define a notion of polynomially bounded.

Definition 4.2.1. *[11, p. 257] We say that a structure $\mathcal{R}$ is* ***polynomially bounded*** *if, for every definable function f, there exists $N \in \mathbb{N}$ such that ultimately $|f(x)| \leq x^N$.*

So, if $\mathcal{R}$ is polynomially bounded then for every definable function, f, there is some x^N which 'grows faster' then f.

4.2.2. Definability of the Exponential Function

Theorem 4.2.2. *[11, p. 257] Let $\mathcal{R}$ be o-minimal and not polynomially bounded. Then the exponential function is definable.*

Proof. Outline Using the monotonicity theorem, for every definable function there is some 'rightmost' interval on which the function is either constant or strictly monotone and continuous. Further, given a result from van den Dries in [12] each definable function is also differentiable on this interval. These results imply that every definable function has a limit at infinity (as the *germs* at $+\infty$ of the definable functions form a Hardy field).

Now, by looking at the limits of definable functions at infinity it is established that there is a function g such that near infinity g' acts like $1/x$. This then implies that the log function is definable. From which it can be easily shown that the exponential function is also definable. □

4.3. General Case of ordered fields

4.3.1. Preliminaries

In this section we let $\mathcal{R}$ be an arbitrary expansion of an ordered field, $\mathcal{R} = \langle R, <, +, -, \cdot, 0, 1, ...\rangle$.

As $\mathcal{R}$ is an ordered field which contains 0, we can consider all the elements of R which are greater than 0 with respect to this ordering. We call these the **positive elements** of R and denote this set by $\text{Pos}(R)$. We also let R^* denote the nonzero elements of R.

Also, due to the field structure, we can think of $(\text{Pos}(R), \cdot, 1)$ as a group with $\cdot$ as the group operation and identity element 1. Below, we will use the notions of group homomorphisms. I will not give the formal definition of these. They can be thought of as maps between groups which preserve the group structure. The following is a result from Pillay and Steinhorn [1, p. 569] which will also be necessary.

Lemma 4.3.1. *Let $\mathcal{G} = (G, +, 0, <)$ be an o-minimal ordered group. Then $\mathcal{G}$ has no proper nontrivial definable subgroups.*

Proof. Omitted. As an outline, we consider a nontrivial definable subgroup, H. As there is a non-zero element contained in H then H must be infinite as o-minimal groups are torsion free, see [3, p. 19]. Hence, by o-minimality H contains an interval. Suppose that H is not equal to G then H contains an interval which is bounded above and below. We suppose that this interval is symmetric about 0 and is the largest such, and from this a contradiction is drawn. See [1, p. 569]. □

4.3.2. Power Boundedness

Definition 4.3.2. *[4, p. 386] A **power function** (of $\mathcal{R}$) is a definable endomorphism of the multiplicative group (Pos$(R), \cdot, 1)$.*

Hence, a power function is a map $f : \text{Pos}(R) \to \text{Pos}(R)$ where for any $x, y \in \text{Pos}(R)$, $f(x \cdot y) = f(x) \cdot f(y)$. It can clearly be seen that this holds for the canonical example of the function $f(x) = x^r$ in the field of real numbers. Also note that for a power function, f, $f'(x) = f'(1) \cdot f(x)/x$.

We now consider the set of all power functions, denoted to be $\mathcal{K}$. Miller shows in his paper that this can be regarded as on ordered field where multiplication is composition, $f \circ g$, addition is pointwise multiplication, $(f+g)(x) = f(x) \cdot g(x)$ and $f < g$ if and only if $f'(1) < g'(1)$ - this works as, importantly, if $f'(1) = g'(1)$ then $f = g$ [4, p. 386].

Define a map $\mathcal{K} \to R$ by $f \mapsto f'(1)$. We call the image of this map the field of exponents of $\mathcal{R}$ and denote it by K [4, p. 386]. (Consider again $f(x) = x^r$ in the field of real numbers; then $f'(1) = k$ i.e. the exponent of x in $f(x)$).

<u>Notation</u>: Let $f \in \mathcal{K}$ and $f'(1) = r \in K$. Then we write x^r to denote f and similarly a^r to denote $f(a)$. Also this notation preserves the usual differentiation for polynomials - $(x^r)' = r \cdot x^{r-1}$. To see this let x^r denote f, then $f'(x) = f'(1) \cdot f(x)/x = r \cdot f(x)/x$. Let $g(x) = f(x)/x$ then

$$g'(x) = \frac{f'(x) \cdot x - f(x)}{x^2}.$$

Hence, $g'(1) = r - 1$ i.e. we denote g by x^{r-1}. Hence $f'(x) = r \cdot x^{r-1}$.

Definition 4.3.3. *[4, p. 387] The structure $\mathcal{R}$ is said to be **power bounded** if for every definable unary function f there exists $r \in K$ such that ultimately $|f(x)| \leq x^r$.*

4.3.3. Exponential Functions

Definition 4.3.4. *[4, p. 389] An **exponential function** for the ordered field R is an ordered group isomorphism*

$$E : (R, <, +, 0) \to (\textit{Pos}(R), <, \cdot, 1).$$

*We say that the structure $\mathcal{R}$ is **exponential** if there exists a exponential function for R definable in $\mathcal{R}$.*

Miller notes that it is an easy exercise to show that if E is differentiable at some point of R then it is differentiable on R and $E'(x) = E'(0)E(x)$. I will show this below.

Let E be differentiable at $c \in R$. Then

$$E'(c) = \lim_{h \to 0} \frac{E(c+h) - E(c)}{h}$$

exists. As E is an isomorphism,

$$E'(c) = \lim_{h\to 0}\frac{E(c)\cdot E(h) - E(c)}{h} = \lim_{h\to 0}E(c)\cdot\frac{E(h)-1}{h}$$

Again as E is an isomorphism $E(0) = 1$. Hence,

$$E'(c) = \lim_{h\to 0}E(c)\cdot\frac{E(h+0)-E(0)}{h} = E(c)\cdot E'(0)$$

This shows that $E'(0)$ exists. Now consider an arbitrary $r \in R$. Then

$$E'(r) = \lim_{h\to 0}\frac{E(r+h)-E(r)}{h} = E(r)\cdot E'(0)$$

as above. Hence E is differentiable on R.

It is given that all unary functions definable in $\mathcal{R}$ are ultimately differentiable [4, p. 386]. From this it follows that if an exponential function is definable in $\mathcal{R}$ then it is differentiable on R. It is also shown by Miller that if some exponential function is definable then there is a definable exponential function E such that $E' = E$ and that this is a **unique** definable function [4, p. 389]. This unique E is sometimes denoted by e^x or $\exp(x)$. Also note that any exponential function E is nonconstant as E is an isomorphism.

The following proposition will be useful later. This is given by Miller as a note, he also mentions that the proof is similar to Proposition 2.3 of his paper (Please note there are two versions of this paper which have slightly different numbering, I am using the version given in the bibliogrpahy).

Proposition 4.3.5. *[4, p. 390] If there exists a definable nonconstant function $L : Pos(R) \to R$ such that $L(xy) = L(x) + L(y)$ for all $x, y \in Pos(R)$ then $\mathcal{R}$ is exponential.*

Proof. First note that the kernel of L, $\ker(L)$, is nonempty as $L(1\cdot 1) = L(1) = L(1) + L(1) \Rightarrow L(1) = 0$. Hence, $\ker(L)$ is a definable subgroup of $(\mathrm{Pos}(R), \cdot, 1)$. By Lemma 4.3.1, $(\mathrm{Pos}(R), \cdot, 1)$ has no proper, nontrivial definable subgroups. Hence $\ker(L)$ is either $\{1\}$ or R. Hence $\ker(L) = \{1\}$ as L is nonconstant. Then via a standard group theoretic result, L is injective. In a similar way it can be shown that L is surjective. Hence L is a bijection.

Consider the bijection $L^{-1} : R \to \text{Pos}(R)$. Let $x, y \in R$. Then for some $a, b \in \text{Pos}(R)$ such that $L(a) = x$ and $L(b) = y$, $L^{-1}(x+y) = L^{-1}(L(a)+L(b)) = L^{-1}(L(ab)) = a \cdot b = L^{-1}(x) \cdot L^{-1}(y)$.

Similarly to the above argument with E, it can be easily shown that L^{-1} is differentiable on R and that $(L^{-1})'(x) = (L^{-1})'(0) \cdot L^{-1}(x)$. Now, L^{-1} is nonconstant $\Rightarrow (L^{-1})'(0) \neq 0 \Rightarrow L^{-1}$ is strictly monotone, however considering the definable function $x \mapsto L^{-1}(x/(L^{-1})'(0))$ we may assume that $(L^{-1})'(0) = 1$ and hence L^{-1} is strictly increasing. Hence, for all $x, y \in R$, $x < y \iff L^{-1}(x) < L^{-1}(y)$.

Hence L^{-1} is an ordered group isomorphism and thus an exponential function for R. Hence $\mathcal{R}$ is exponential. □

The main theorem of this chapter is that $\mathcal{R}$ is either exponential or power bounded. However, without introducing any more complex ideas it is possible to show that this 'or' must be exclusive. This is the result of the following proposition.

Proposition 4.3.6. *[4, p. 390] If $\mathcal{R}$ is exponential, then $\mathcal{R}$ is not power bounded.*

Proof. This proof is given by Miller but I will add some more detail. First we show that for all $r \in R$, $\lim_{x\to+\infty}(e^x/x^r) = +\infty$.

Consider

$$(\frac{e^x}{x^r})' = \frac{e^x \cdot x^r - r \cdot x^{r-1} \cdot e^x}{x^{2r}} = e^x \cdot x^{-r} - r \cdot x^{-r-1} \cdot e^x = e^x \cdot x^{-r}(1 - r \cdot x^{-1}).$$

For $r \leq 0$ this is clearly positive for positive x. If $r > 0$ then this is positive for $x > r$. Hence e^x/x^r is strictly increasing. Suppose that e^x/x^r is bounded, i.e. $\lim_{x\to+\infty} e^x/x^r = c$ for some $c \in \text{Pos}(R)$. Then

$$c = \lim_{x\to+\infty} \frac{e^{x+1}}{(x+1)^r} = \lim_{x\to+\infty} \frac{e^1 \cdot e^x}{x^r \cdot (1+x^{-1})^r} = e^1 \cdot c.$$

Hence $e^1 = 1$. However, $e^0 = 1$ and e is strictly increasing so this is a contradiction. Hence, e^x/x^r is strictly increasing and not bounded for any $r \in R$. So, $\lim_{x\to+\infty} e^x/x^r = +\infty$ for all $r \in R$. Hence $\mathcal{R}$ is not power bounded. □

4.3.4. The Growth Dichotomy

I will give an outline of the proof of the growth dichotomy. It will first be necessary to define the functions P_f and v and give some properties.

Definition 4.3.7. *[4, p. 388] Let f be a definable unary function which is ultimately nonzero. Then the definable set*

$$\{t \in \mathit{Pos}(R) : \mathit{lim}_{x\to+\infty}(f(tx)/f(x)) \in \mathit{Pos}(R)\}$$

is a subgroup of $(\mathit{Pos}(R), \cdot, 1)$ and so, by Lemma 4.3.1, is either $\{1\}$ or $\mathit{Pos}(R)$. If this subgroup is $\mathit{Pos}(R)$ then we define $P_f : \mathit{Pos}(R) \to \mathit{Pos}(R)$ by

$$P_f(t) := \mathit{lim}_{x\to+\infty}(f(tx)/f(x)).$$

Importantly, note that for all $s, t \in \text{Pos}(R)$, $P_f(st) = P_f(s) \cdot P_f(t)$. Hence, P_f is an endomorphism of $(\text{Pos}(R), \cdot, 1)$ and so is a power function, x^r for some $r \in R$. Clearly, if f is a power function x^r then $P_f = f$. Hence, $P_{P_f} = P_f = f$, moreover, $P(P_f/f) = P(f/f) = 1$ (Note that we write 1 here to mean the constant function taking elements from Pos(R) to $1 \in \text{Pos}(R)$).

Now, for the sake of brevity I will not go into much detail regarding the idea of *germs*. The germ of a function, f, at $+\infty$ is an equivalence class of f such that $f \sim g \iff f$ and g are ultimately the same. Let H denote the set of all germs of definable unary functions and H^* be the nonzero elements of H. We also write f to mean the germ of f. We will write x to denote the identity function (it should be clear from the context when it is meant as a function or a variable), $1/f$ to denote the multiplicative inverse of f and f^{-1} to denote the compositional inverse of f.

We now define a valuation v on H^* which has the property that for any $f \in H^*$ $v(f) = 0$ if $\lim_{x\to+\infty} f(x) \in R^*$, $v(f) > 0$ if $\lim_{x\to+\infty} f(x) = 0$ and $v(f) < 0$ if $\lim_{x\to+\infty}|f(x)| = +\infty$. Also, for any $f, g \in H^*$ if $v(f) = v(g)$ then $\lim_{x\to+\infty}(f(x)/g(x)) = c$ for some $c \in R^*$, if $c = 1$ then we write $f \sim g$. We also use $+$ to denote the group operation in $v(H^*)$ and $-$ to denote the group inverse in $v(H^*)$ hence, $v(fg) = v(f) + v(g)$ and $v(f) - v(g) = v(f/g)$.

We say that $f \in H^*$ is **infinitely increasing** if $\lim_{x\to+\infty} f(x) = +\infty$.

Theorem 4.3.8. *[4, p. 393] Either $\mathcal{R}$ is exponential or $\mathcal{R}$ is power bounded.*

Proof. Outline We split into two cases.

Case 1 There exists $f \in H^*$ with $v(f) \neq 0$ and $v(f'/f) \neq v(1/x)$.

First, we may suppose that f is infinitely increasing. (As if not, then exactly one of $1/f$, $-f$ or $-1/f$ is infinitely increasing [4, p. 391], call this g. Also, $|v(f)| = |v(g)|$ [4, p. 391], hence $v(g) \neq 0$ and then $v(f'/f) = v(g'/g)$ by Proposition 3.3 of Miller's paper [4, p. 392]. Hence $v(g'/g) \neq v(1/x)$ so we may continue the proof with g in place of f, with g infinitely increasing). Now, for large enough x, f has an inverse, denoted f^{-1} (As, due to o-minimality f must be strictly increasing on some interval $(a, +\infty)$).

We may assume $v(f'/f) < v(1/x)$, replacing f with f^{-1} if not. To see why this is the case suppose that $v(f'/f) > v(1/x)$. Then

$$v(f'/f) - v(1/x) = v(f'/f) + v(x) = v(xf'/f) > 0$$

$$\Rightarrow \lim_{x \to +\infty} xf'(x)/f(x) = 0.$$

Similary if $\lim_{x \to +\infty} f^{-1}(x)/xf^{-1\prime}(x) = 0$ then $v(f^{-1\prime}/f^{-1}) < v(1/x)$ as required. To see this note that

$$\lim_{x \to +\infty} f^{-1}(x)/xf^{-1\prime}(x)$$

$$= \lim_{x \to +\infty} f'(f^{-1}(x))f^{-1}(x)/x$$

$$= \lim_{x \to +\infty} f'(f^{-1}(f(x)))f^{-1}(f(x))/f(x) \text{ (as } f \text{ is infinitely increasing)}$$

$$= \lim_{x \to +\infty} xf'(x)/f(x) = 0.$$

Now, from Proposition 3.4 of Miller's paper [4, p. 392], there is some $h \in H^*$ with $h' \sim f'/f$. For some intuition about why such a h' is important notice that this is saying that h' is equivalent to the derivative of 'log' $f(x)$, I put apostrophes around the log here to emphazise that we don't actually yet have a formally defined notion of the log function in this structure and this is just mentioned to provide some intuition. Then $(h \circ f^{-1})' \sim 1/x$ i.e. the derivative of $h \circ f^{-1}$ is in some sense equivalent to the derivative of 'log' x.

To see that $(h \circ f^{-1})' \sim 1/x$ we must show that $\lim_{x \to +\infty} x(h \circ f^{-1})'(x) = 1$. Note that

$(h \circ f^{-1})' = h' \circ f^{-1} \cdot f^{-1\prime} = h' \circ f^{-1}/f' \circ f^{-1}$. Hence

$$\lim_{x\to+\infty} x(h \circ f^{-1})'(x)$$

$$= \lim_{x\to+\infty} x \cdot h'(f^{-1}(x))/f'(f^{-1}(x))$$

$$= \lim_{x\to+\infty} xf'(f^{-1}(x))/f(f^{-1}(x)) \cdot f'(f^{-1}(x)) \text{ (as } h' \sim f'/f \text{)}$$

$$= 1.$$

Let $g : h \circ f^{-1}$, so $g' \sim 1/x$.

Now, as the mean value theorem holds for definable functions [4, p. 386], we can apply this to $g(x)$. Let $t \in \text{Pos}(R) - \{1\}$. For $t > 1$, we get

$$g'(c) = \frac{g(xt) - g(x)}{xt - x}$$

for some c with $x < c < xt$. Combined with the fact that $g' \sim 1/x$, we get that ultimately $g(xt)-g(x)$ is positive and bounded above and below in R, see [11, p. 258] for details. As g is definable, $g(xt) - g(x)$ is clearly definable and hence has a limit in $R \cup \{-\infty, +\infty\}$ [11, p. 386], hence has a limit in R^*. For $t \in (0, 1)$, we apply the mean value theorem similary to $g(x/t) - g(x)$ to see that

$$\lim_{x\to+\infty} g(x/t) - g(x) = -\lim_{x\to+\infty} g(xt) - g(x) \in R^*.$$

Hence, the function $G : \text{Pos}(R) \to R$ defined by

$$G(t) := \lim_{x\to+\infty}(g(tx) - g(x))$$

is well-defined and is clearly a definable function with $G(t) \neq 0$ for all $t \neq 1$.

$G(1) = 0$ so G is nonconstant. Also, $G(st) = G(s) + G(t)$ for all $s, t \in \text{Pos}(R)$. (To see this: Let $\epsilon > 0$ then there is $\delta_1, \delta_2 > 0$ such that

$$x > \delta_1 \Rightarrow g(tx) - g(x) - G(t) < \epsilon/2$$

and

$$x > \delta_2 \Rightarrow g(sx) - g(x) - G(s) < \epsilon/2.$$

Let $\delta = \max\{\delta_1/2, \delta_2\}$ and let $x > \delta$, then $s \cdot x > \delta_1$ so $g(stx) - g(sx) - G(t) < \epsilon/2$ (1) and $x > \delta_2$ so $g(sx) - g(x) - G(s) < \epsilon/2 \Rightarrow -g(x) - G(s) - \epsilon/2 < -g(sx)$ (2). Substituting (2) into (1) gives

$$g(stx) - g(x) - G(s) - \epsilon/2 - G(t) < \epsilon/2 \Rightarrow g(stx) - g(x) - G(s) - G(t) < \epsilon).$$

Hence by Proposition 4.3.5, $\mathcal{R}$ is exponential.

<u>Case 2</u> For all $f \in H^*$, if $v(f'/f) \neq v(1/x)$ then $v(f) = 0$. It is first shown that for all $f \in H^*$, P_f exists.

If $v(f) = 0$ then $\lim_{x\to+\infty} f(x) \in R^* \Rightarrow \lim_{x\to+\infty} f(tx)/f(x) = 1$ for all $t \in \text{Pos}(R)$, hence P_f is the constant function 1.

For the case that $v(f) \neq 0$ we consider the map $g(x) = f(2x)/f(x)$. If we show that $v(g) = 0$ then P_2 is defined and hence P_f is defined for all $t \in \text{Pos}(R)$ (from Lemma 4.3.1 and the definition of P_f). From the assumption of this case, if we show that $v(g'/g) \neq v(1/x)$ then $v(g) = 0$. Now, $v(f) \neq 0$, hence by case assumption $v(xf'/f) = 0$ i.e. $\lim_{x\to+\infty} xf'(x)/f(x) \in R^*$. Hence

$$\lim_{x\to+\infty} xf'(x)/f(x) = \lim_{x\to+\infty} 2xf'(2x)/f(2x) \in R^*$$

and

$$xg'(x)/g(x) = 2xf'(2x)/f(2x) - xf'(x)/f(x),$$

so $\lim_{x\to+\infty} xg'(x)/g(x) = 0$ i.e. $v(g'/g) \neq v(1/x)$.

Hence for all $f \in H^*$, $P_f = x^r$ for some $r \in K$. We now show that $v(f) = v(P_f)$.

Consider the case where $P_f \neq 1$. Well, as seen above $P(P_f/f) = 1$ and if we show that $v(P_f/f) = 0$ then $v(f) = v(P_f)$. So we may assume that $P_f = 1$ and show that $v(f) = 0$ i.e. $v(f) = v(P_f)$. This is done again by using the mean value theorem. We get that

$$f'(c) = \frac{f(2x) - f(x)}{2x - x}$$

for some c with $x < c < 2x$, using this Miller shows that $v(f'/f) \neq v(1/x)$ and hence by the case assumption $v(f) = 0$, to see full details see [4, p. 394].

This concludes the proof since $v(f) = v(P_f) \Rightarrow \lim_{x\to+\infty} f(x)/x^r = c$ for some $c \in R^*$, hence for all suffciently large x, $|f(x)| \leq x^{r+1}$. Hence $\mathcal{R}$ is power bounded. $\square$

5. Brief Discussion of Applications

5.1. Introduction

O-minimality has many applications across various areas of mathematics. This chapter is intended to give a brief look at a few such applications, providing a list which is by no means exhaustive. The nature of these applications is that they are rather complex, so due to restriction on time I am not able to go into much detail. The intention is really just to show that o-minimality is very applicable outside of model theory.

5.2. Vapnik-Chervonenkis Dimension

I will first give the formal definition of Vapnik-Chervonenkis Dimension taken from [13, p. 327]. Let D be a given set. For every $X \subseteq D$ we define a **dichotomy** on X as a function $c : X \to \{0, 1\}$. We then say that a function $f : D \to \mathbb{R}$ **implements** the dichotomy c if and only if $c(x) > 0 \iff f(x) > 0$. Let F be any class of functions. We say that a subset $X \subseteq D$ is **shattered** by F if each dichotomy on X can be implemented by some $f \in F$. The Vapnik-Chervonenkis (VC) dimension of F, $VC(F)$, is then defined to be the (possibly infinite) supremum of the set of integers k where there is some subset $X \subseteq D$ of cardinality k that can be shattered by F.

To get a more intuitive understanding of the notions defined above we will consider an example from neural networks.

Consider a set of data points on a 2-D plane, this is our subset X, then F shatters X if however we label the data points (using 0 and 1) there is some function in F which draws a dividing line and separates all 1 points from 0 points. $VC(F)$ is then the largest cardinality of a subset X such that this is the case. The VC dimension is a measure of the complexity of a neural network and indicates the network's ability to do genuine learning.

(In the example of a neural network, D is the set of all inputs, X is the training set, c is in some sense the supervised learning process and a function f implements c if it gives the same answers as c on the training set X).

How is this related to o-minimality? Well, consider a structure $A = \langle R, ...\rangle$ in a language $\mathcal{L}$ and some $\mathcal{L}$-formula, $\theta(w_1, ..., w_r, x_1, ..., x_m)$. We then define a class of functions $F = \{\theta_{\bar{w}} : \bar{w} \in \mathbb{R}^r\}$ mapping $\mathbb{R}^m \to \{0, 1\}$ by $\theta_{\bar{w}}(x_1, ..., x_m) = 1 \iff \theta(\bar{w}, x_1, ..., x_m)$ holds in A.

It was shown that if A is o-minimal then $VC(F) < \infty$ [14, p. 383]. (1)

Further, formulas in appropriate languages can be used to describe certain neural networks. Hence, we can deduce that the VC dimension of such neural networks is finite.

Also, the property in (1) is also known as the dependence property or the non-independence property, NIP, which has several applications in model theory and has become a more active area of research over the past couple of decades, see [15].

5.3. Number Theory

In [16], Pila and Wilkie consider the density of rational points in given subsets of $\mathbb{R}^n$. A *rational point* in $\mathbb{R}^n$ is simply a point with rational coordinates. As discussed in this paper, results had already been given saying that on certain sets there is in some sense only few rational points. However, Pila and Wilkie extend this by looking at arbitrary definable sets in o-minimal structures. Giving the result that there are ‘few’ rational points in any definable set. Clearly in most definable subsets of $\mathbb{R}^n$ there is an infinite number of rational points, hence we restrict attention to rational points where the numerator and denominator are bounded and we also remove the *algebraic part* of the subset. This result has had several consequences in number theory.

5.4. Algebraic Geometry

Let $\mathcal{R}$ be the ordered field of real numbers. Then the sets definable in $\mathcal{R}$ using constants are exactly the semialgebraic sets, defined in 1.4.7. One direction of this was shown in 1.3.9. The other direction results from the *Tarksi-Seidenberg Theorem* which essentially gives quantifier elimination for $\mathcal{R}$. This then shows that $\mathcal{R}$ is o-minimal as the semialgebraic sets on the real line are all the finite unions of intervals and points [3, p. 37].

The semialgebraic sets are the main focus of study in algebraic geometry, hence conclusions drawn about definable sets in o-minimal structures are of considerable interest.

Appendices

A. Aside on Topology

Definition A.1. *[17, p. 28] Given a set, X, a collection τ of subsets of X is a **topology** for X if it has the following properties:*

(i) $\emptyset \in \tau$, $X \in \tau$

(ii) *if* $U_1, U_2 \in \tau$, *then* $U_1 \cap U_2 \in \tau$

(iii) *if* $U_\lambda \in \tau$ *for all* $\lambda \in \Delta$ *for some indexing set* Δ, *then* $\bigcup_{\lambda \in \Delta} U_\lambda \in \tau$

We call (X, τ) a topological space and the subsets of τ are called the open sets of X.

Definition A.2. *[17, p. 29] Given a topological space, (X, τ), and a subset $A \in X$, we say that A is **closed** if its complement is open.*

Definition A.3. *[18, p. 60] If X is a topological space and p is a point in X, a **neighbourhood** of p is a subset V of X that includes an open set U containing p, $p \in U \subseteq V$.*

Definition A.4. *[18, p. 61] Let S be a subset of a topological space X. A point p in X is a **limit point** of S if every neighbourhood of p contains at least one point of S which is not p itself.*

Definition A.5. *[18, p. 61] The **closure** of a set S is the set S together with all of its limit points.*

Definition A.6. *[18, p. 63] Given a topological space, (X, τ), and a subset $A \in X$, we say that A is **dense** in A if the only closed subset of X containing A is X.*

Remark A.7. The following result will be useful to note. Given a topological space, (X, τ), and a subset $A \in X$, if A is dense in X then for any open subset $C \subseteq X$, $C \cap A \neq \emptyset$. Here is a short proof. Suppose $C \cap A = \emptyset$. Then $A \subseteq X - C$, which is a proper closed subset of X. Contradiction.

Definition A.8. *[17, p. 32] A function $f : X \to Y$ between two topological spaces X and Y is **continuous** if for every open set, V, of Y. the inverse image of V, $f^{-1}(V)$, is open in X.*

Definition A.9. *[17, p. 39] A topological space X is **Hausdorff** if for each distinct pair of points $x, y \in X$ there exists open sets U, V in X such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.*

Remark A.10. [3, p. 95] The following result will be useful to note. Given a continuous map $f : X \to Y$ from a topological space X to a hausdorff space Y, the graph $\Gamma(f)$ is a closed subset of $X \times Y$.

Definition A.11. *[18] Suppose X,Y are topological spaces with Y a hausdorff space. Let p be a limit point of $S \subseteq X$ and $L \in Y$. For a function $f : S \to Y$, it is said that the **limit of** f **as** x **approaches** p **is L** and write $\lim_{x \to p} f(x) = L$ if the following holds:*

For every open neighbourhood V of L, there exists an open neighbourhood U of p such that $f(U \cap S - \{p\}) \subseteq V$.

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